Midterm 1

Friday, October 27, 2023

Use your own notebook paper, or get some from me. Feel free to do some scratch work that you don't turn in before writing the solution that you do turn in. I encourage you to write in pencil. There are 38 points in total.

1. (a) (3 points) Let (X, d) be a metric space. Define what it means for a sequence $p_1, p_2, \ldots$ in X to converge to a limit $\ell \in X$.

   (b) (5 points) Suppose that $p_n \to \ell$ and $p_n \to \ell'$. Prove that $\ell = \ell'$.
   Hint: Prove that $d(\ell, \ell') < \epsilon$ for every $\epsilon > 0$, using the triangle inequality.

2. Let (X, d) be a metric space. Our notation for the *open ball* of radius r around a point $p \in X$ is

$$B_r(p) = \{q \in X : d(p, q) < r\}.$$

   Now define the *closed ball*

$$\bar{B}_r(p) = \{q \in X : d(p, q) \leq r\}.$$

   (a) (3 points) Define what it means for a subset $A \subset X$ to be open.

   (b) (3 points) Define what it means for a subset $A \subset X$ to be closed.

   (c) (5 points) In lecture we saw that $B_r(p)$ is open. Now prove that the $\bar{B}_r(p)$ is closed.
   Hint: You could prove it directly, or you could prove that the complement of $\bar{B}_r(p)$ is open.

(d) (5 points) Define what it means for a point $p \in X$ to be in the closure of a subset $A \subset X$. State an equivalent criterion for p to be in the closure of A that was proved in lecture.

(e) (3 points) Prove that the closure $\overline{B_r(p)}$ is contained in $\bar{B}_r(p)$. You may use anything proved on the homework.

(f) (3 points) Give an example to show the inclusion of part (e) could be strict, that is, $\overline{B_r(p)}$ could be a proper subset of $\bar{B}_r(p)$. Hint: You might take X to be a set with a discrete metric, or $X = \mathbb{Z}$ with the usual metric inherited from $\mathbb{R}$.

3. (a) (3 points) Let (X, d_X) and (Y, d_Y) be metric spaces. Define what it means for a function $f\colon X \to Y$ to be continuous.

(b) (5 points) Let (X, d) be a metric space. Given two functions $f, g\colon X \to \mathbb{R}$, define a function $h\colon X \to \mathbb{R}^2$ by $h(p) = (f(p), g(p))$. Prove that if f and g are continuous with respect to the usual metric on $\mathbb{R}$, then h is continuous with respect to the Euclidean or taxicab or square metric on $\mathbb{R}^2$ (your choice).