Midterm 2

Friday, November 17, 2023

There are 38 points in total.

1. Let $f\colon X \to Y$, let $A \subset X$, and let $B \subset Y$.
   (a) (3 points) Prove that $f(A \cap f^{-1}(B)) = f(A) \cap B$.
   (b) (3 points) Give an example where $f(A \cup f^{-1}(B))$ is different from $f(A) \cup B$.
2. Let X and Y be topological spaces.
   (a) (3 points) Define what it means for a map $f\colon X \to Y$ to be continuous.
   (b) (5 points) Let X be Sierpiński space: that is, $X = \{0, 1\}$, and the open sets are $\varnothing$, $\{1\}$, and X. Prove that if every one-point subset of Y is closed, then every continuous map $f\colon X \to Y$ is constant.
   (c) (5 points) Let X be Sierpiński space again, and let $Y = \mathbb{R}$ with the lower semi-continuous topology, whose open sets are $\varnothing$, $\mathbb{R}$, and intervals of the form (a, ∞). Give an example of a continuous map $f\colon X \to Y$ that is not constant.
3. Let X and Y be topological spaces.
   (a) (3 points) Define the product topology on $X \times Y$.
   (b) (5 points) Prove that if $F \subset X$ and $G \subset Y$ are closed, then $F \times G$ is closed in $X \times Y$.
   (c) (3 points) Let $q\colon X \times Y \to Y$ be the projection defined by $q(x, y) = y$. Prove that q is continuous.
   (d) (5 points) Prove that if $W \subset X \times Y$ is open, then $q(W) \subset Y$ is open.
   (e) (3 points) Give a counterexample to the claim that if $F \subset X \times Y$ is closed, then $q(F) \subset Y$ is closed.
   Hint: The claim is true if X is compact, so choose an example where X is not compact.