Final Exam

Tuesday, December 10, 2024

Use your own notebook paper, or get some from me. If you get stuck on one of the earlier parts, you can still use it in the later parts. There are NN points in total.

1. (a) (3 points) State the Baire category theorem, either for complete metric spaces or for locally compact Hausdorff spaces.
   (In either case there are two good answers.)

   (b) (3 points) A point p in a topological space X is *isolated* if the one-point set $\{p\}$ is open. Give an example of a topological space with at least one isolated point and at least one non-isolated point.

   (c) (3 points) Do whichever one that goes best with your answer to part (a):

   (c1) Prove that $p \in X$ is not isolated if and only if $X \setminus \{p\}$ is dense in X.

   (c2) Prove that $p \in X$ is not isolated and only if and only if the one-point set $\{p\}$ is nowhere dense.

   (d) (3 points) Let X be a non-empty topological space with no isolated points. Use the Baire category theorem to prove that if X admits a complete metric, or is locally compact and Hausdorff, then X is uncountable.

2. The point of this problem is to prove that $X \times Y$ is Hausdorff if and only if X is Hausdorff and Y is Hausdorff.

   (a) (3 points) Define the product topology: if X and Y are topological spaces, then a subset $W \subset X \times Y$ is open if and only if ...
   (Fill in the blank. There are two good answers.)

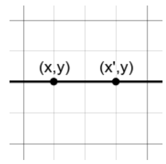
   (b) (3 points) Define what it means for a topological space X to be Hausdorff.

   (c) (4 points) Give an example of a topology on $\mathbb{R}$ that is Hausdorff, and a topology on $\mathbb{R}$ that is not Hausdorff.

   (d) (3 points) Let X and Y be Hausdorff spaces. Prove that the product topology on $X \times Y$ is Hausdorff.

   (e) (3 points) Let X and Y be non-empty topological spaces, and suppose that the product topology on $X \times Y$ is Hausdorff. Prove that X is Hausdorff.

Hint: Given two distinct points $x, x' \in X$, choose a point $y \in Y$, and consider the points (x, y) and (x', y) in $X \times Y$.



(f) (1 point) Write "Similarly, if $X \times Y$ is Hausdorff then Y is Hausdorff."

3. Let (X, d) be a metric space. A map $f\colon X \to X$ is called a *weak contraction mapping* if

$$d(f(x), f(y)) < d(x, y)$$

for all $x, y \in X$, while a *contraction mapping* if there is an $r \in [0, 1)$ such that

$$d(f(x), f(y)) \leq r \cdot d(x, y)$$

for all $x, y \in X$. Earlier in the term we proved the Banach fixed point theorem: if X is complete and $f\colon X \to X$ is a contraction mapping, then there is a unique point $x \in X$ such that $f(x) = x$. This problem asks you to prove that if X is compact, then any weak contraction mapping is a contraction mapping.

(a) (3 points) Let Y be a topological space. Define an *open over* of a subset $A \subset Y$, and a *subcover*. Define what it means for A to be *compact*.

(b) (3 points) Let X be a metric space, let $f\colon X \to X$ be continuous, and let $s \geq 0$. Prove that the set

$$U_s = \{(x, y) \in X \times X : d(f(x), f(y)) < s \cdot d(x, y)\}$$

is open in $X \times X$. You may assume without proof that the metric d is continuous as a map from $X \times X$ with the product topology to $\mathbb{R}$ with the usual topology.

Hint: Cook up a map $F\colon X \times X \to \mathbb{R}^2$ such that U_s is the preimage of the open set

$$V_s = \{(z, w) \in \mathbb{R}^2 : z < s \cdot w\},$$

then prove that F is continuous.

(c) (3 points) Let X be a compact metric space, and let $f\colon X \to X$ be a weak contraction mapping. Prove that f is actually a contraction mapping.

Hint: Consider the open cover of $X \times X$ given by U_s for all $s \in [0, 1)$.