Solutions to Final Exam

1. (a) (3 points) State the Baire category theorem, either for complete metric spaces or for locally compact Hausdorff spaces.

   (In either case there are two good answers.)

   Solution: A countable union of nowhere dense sets has empty interior, or equivalently, a countable intersection of dense, open sets is again dense.

   (b) (3 points) A point p in a topological space X is *isolated* if the one-point set $\{p\}$ is open. Give an example of a topological space with at least one isolated point and at least one non-isolated point.

   Solution: Of course there are many possible answers. You might take $[0, 1] \cup \{2, 3, 4, \dots\}$ with the subspace topology from the usual topology on $\mathbb{R}$: the points $2, 3, 4, \dots$ are all isolated, while the points in $[0, 1]$ are not. Or you might take

   $$W = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots, 0\},$$

   again with the subspace topology from the usual topology on $\mathbb{R}$: every point is isolated except for 0.

   (c) (3 points) Do whichever one that goes best with your answer to part (a):

   (c1) Prove that $p \in X$ is not isolated if and only if $X \setminus \{p\}$ is dense in X.
   Solution: By definition, $X \setminus \{p\}$ is dense if and only if its closure is all of X. The closure is the smallest closed set containing $X \setminus \{p\}$, and the only two sets containing $X \setminus \{p\}$ are $X \setminus \{p\}$ itself and the whole space X, which is always closed. Thus the closure is X if and only if $X \setminus \{p\}$ is not closed, which is true if and only if $\{p\}$ is not open.
   Alternatively, $X \setminus \{p\}$ is dense if and only if it intersects every non-empty open set. The only non-empty set that it doesn't intersect is $\{p\}$, so it's dense if and only if $\{p\}$ is not open.

(c2) Prove that $p \in X$ is not isolated and only if and only if the one-point set $\{p\}$ is nowhere dense.

Solution: By definition, $\{p\}$ is nowhere dense if and only if its interior is empty. The interior is the biggest open set contained in $\{p\}$, and the only two sets contained in $\{p\}$ are $\{p\}$ itself and $\varnothing$, which is always open. Thus the interior is $\varnothing$ if and only if $\{p\}$ is not open.

(d) (3 points) Let X be a non-empty topological space with no isolated points. Use the Baire category theorem to prove that if X admits a complete metric, or is locally compact and Hausdorff, then X is uncountable.

Solution: Depending on your answer to parts (a) and (c), you'll prefer one of these two:

We can write $X = \bigcup_{p \in X} \{p\}$. Because X has no isolated points, each one-point set $\{p\}$ is nowhere dense by part (c1). If X were countable then the Baire category theorem would say that the interior of X is empty; but X is open, so its interior is X, which we have assumed is not empty.

Alternatively, we see that $\bigcap_{p \in X} (X \setminus \{p\}) = \varnothing$. Because X has no isolated points, each $X \setminus \{p\}$ is dense in X, and because we're in a Hausdorff space, points are closed, so $X \setminus \{p\}$ is open. If X were countable then the Baire category theorem would say that this intersection of open dense sets is dense; but if X is not empty then $\varnothing$ is not dense in X.

2. The point of this problem is to prove that $X \times Y$ is Hausdorff if and only if X is Hausdorff and Y is Hausdorff.

(a) (3 points) Define the product topology: if X and Y are topological spaces, then a subset $W \subset X \times Y$ is open if and only if ...

(Fill in the blank. There are two good answers.)

Solution: You could either say that $W = \bigcup_{i \in I} U_i \times V_i$ for some open sets $U_i \subset X$ and $V_i \subset Y$, or that for every $(x, y) \in W$ there are open sets $U \subset X$ and $V \subset Y$ with $(x, y) \in U \times V \subset X \times Y$.

(b) (3 points) Define what it means for a topological space X to be Hausdorff.

Solution: Distinct points have disjoint neighborhoods: for any two points $p, q \in X$ with $p \neq q$, there are open sets $U, V \subset X$ with $p \in U$, $q \in V$, and $U \cap V = \varnothing$.

(c) (4 points) Give an example of a topology on $\mathbb{R}$ that is Hausdorff, and a topology on $\mathbb{R}$ that is not Hausdorff.

Solution: You only need two examples, but I'll run through all the topologies we've seen on $\mathbb{R}$.

Hausdorff:

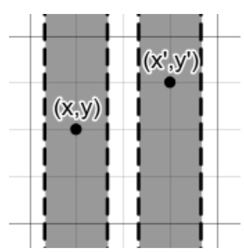
- The usual topology, coming from the metric $d(x,y) = |x-y|$.
- The lower limit topology, where the open subsets are unions of half-open intervals $[a,b)$.
- The discrete topology, where every subset is open.

Not Hausdorff:

- The lower semi-continuous topology, where the open sets are $\varnothing$, $\mathbb{R}$, and (a,∞) for any $a \in \mathbb{R}$.
- The finite complement topology, where $U \subset \mathbb{R}$ is open if and only if $\mathbb{R} \setminus U$ is finite or $U = \varnothing$.
- The particular point topology, where $U \subset \mathbb{R}$ is open if and only if $0 \in U$ or $U = \varnothing$.
- The indiscrete topology: the only open sets are $\varnothing$ and $\mathbb{R}$.

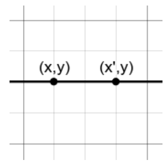
(d) (3 points) Let X and Y be Hausdorff spaces. Prove that the product topology on $X \times Y$ is Hausdorff.

Solution: Let (x,y) and (x',y') be points of $X \times Y$. If $(x,y) \neq (x',y')$, then either $x \neq x'$ or $y \neq y'$ (or both). If $x \neq x'$, choose open sets $U, U' \subset X$ such that $x \in U$, $x' \in U'$, and $U \cap U' = \varnothing$. Then $U \times Y$ and $U' \times Y$ are disjoint neighborhood of (x,y) and (x',y') in $X \times Y$. If $y \neq y'$ then the argument is similar.



(e) (3 points) Let X and Y be non-empty topological spaces, and suppose that the product topology on $X \times Y$ is Hausdorff. Prove that X is Hausdorff.

Hint: Given two distinct points $x, x' \in X$, choose a point $y \in Y$, and consider the points (x, y) and (x', y) in $X \times Y$.



Solution: Given two distinct points $x, x' \in X$, choose a point $y \in Y$, which is possible because we assumed that Y is not empty, and consider the points (x, y) and (x', y) in $X \times Y$. Because $X \times Y$ is Hausdorff, there are open sets $W, W' \subset X \times Y$ such that $(x, y) \in W$, $(x', y) \in W'$, and $W \cap W' = \varnothing$. By the definition of the product topology, there are open sets $U, U' \subset X$ and $V, V' \subset Y$ such that $(x, y) \in U \times V \subset W$ and $(x', y) \in U' \times V' \subset W'$. Thus $x \in U$, $x' \in U'$, and $U \cap U' = \varnothing$; to justify the last assertion, you could say that if $x'' \in U \cap U'$ then (x'', y) is in both $U \times V$ and $U' \times V'$, thus in both W and W', which is impossible because $W \cap W' = \varnothing$.

(f) (1 point) Write "Similarly, if $X \times Y$ is Hausdorff then Y is Hausdorff."

Solution: Similarly, if $X \times Y$ is Hausdorff then Y is Hausdorff.

3. Let (X, d) be a metric space. A map $f: X \to X$ is called a *weak contraction mapping* if

$$d(f(x), f(y)) < d(x, y)$$

for all $x, y \in X$, while a *contraction mapping* if there is an $r \in [0, 1)$ such that

$$d(f(x), f(y)) \leq r \cdot d(x, y)$$

for all $x, y \in X$. Earlier in the term we proved the Banach fixed point theorem: if X is complete and $f: X \to X$ is a contraction mapping, then there is a unique point $x \in X$ such that $f(x) = x$. This problem asks you to prove that if X is compact, then any weak contraction mapping is a contraction mapping.

(a) (3 points) Let Y be a topological space. Define an *open over* of a subset $A \subset Y$, and a *subcover*. Define what it means for A to be *compact*.

Solution: An *open cover* of A is a collection of open sets $U_i \subset X$, indexed by a set I, whose union $\bigcup_{i \in I} U_i$ contains A. Extracting *subcover* means choosing a subset $J \subset I$ such that the union $\bigcup_{j \in J} U_j$ still contains A. To say that A is compact means that every open cover has a finite subcover.

(b) (3 points) Let X be a metric space, let $f\colon X \to X$ be continuous, and let $s \geq 0$. Prove that the set

$$U_s = \{(x, y) \in X \times X : d(f(x), f(y)) < s \cdot d(x, y)\}$$

is open in $X \times X$. You may assume without proof that the metric d is continuous as a map from $X \times X$ with the product topology to $\mathbb{R}$ with the usual topology.

Hint: Cook up a map $F\colon X \times X \to \mathbb{R}^2$ such that U_s is the preimage of the open set

$$V_s = \{(z, w) \in \mathbb{R}^2 : z < s \cdot w\},$$

then prove that F is continuous.

Solution: Take the map $F\colon X \times X \to \mathbb{R}^2$ given by

$$F(x, y) = (d(f(x), f(y)), d(x, y)).$$

If $V_s \subset \mathbb{R}^2$ is the open subset suggested in the hint, then we see that $U_s = F^{-1}(V_s)$, so if we prove that F is continuous then U_s will be open.

There are several ways to argue that F is continuous; here is a very terse one. We have seen that a map into a product is continuous if and only if its components are continuous, and that the projections $p, q\colon X \times X \to X$ onto the first and second factors are continuous, and we are assuming that $d\colon X \times X \to \mathbb{R}$ is continuous. The second factor of F is d, and the first is $d \circ ((f \circ p) \times (f \circ q))$.

(c) (3 points) Let X be a compact metric space, and let $f\colon X \to X$ be a weak contraction mapping. Prove that f is actually a contraction mapping.

Hint: Consider the open cover of $X \times X$ given by U_s for all $s \in [0, 1)$.

Solution: Perhaps we should argue that f is continuous, but that's not hard: take $\delta = \epsilon$. I don't mind if you skip this step.

Now because f is a weak contraction mapping, we see that

$$X \times X = \bigcup_{s \in [0,1)} U_s.$$

Because X is compact, the product $X \times X$ is compact, so we can extract a finite subcover: that is, there are $s_1, \ldots, s_n \in [0, 1)$ such that $X \times X = U_{s_1} \cup \cdots \cup U_{s_n}$. Let $r = \max(s_1, \ldots, s_n)$, and observe that $r < 1$. Then $X \times X = U_r$, so we have

$$d(f(x), f(y)) < r \cdot d(x, y)$$

for all x and y in X, which is stronger than what we need.