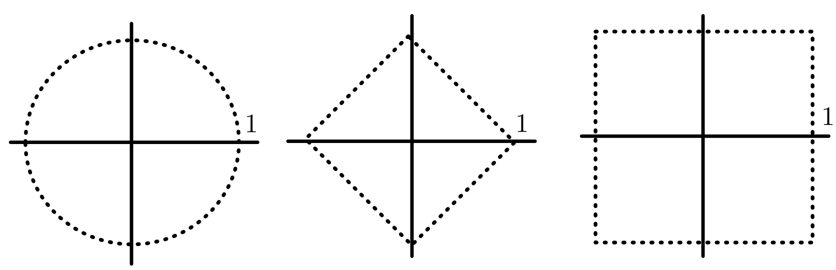
# Solutions to Homework 1

1.1. (a) For each of the three metrics in Example 1.4, sketch the open ball of some radius $r > 0$ around the origin in $\mathbb{R}^2$:

$$B_r(0) = \{(x, y) \in \mathbb{R}^2 : d((x, y),\, 0) < r\}.$$

**Solution:**

$d_2$: $\qquad$ $d_1$: $\qquad$ $d_\infty$:




(b) For one of the three metrics (your choice), prove or give a counterexample to the following statement: a sequence of points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $\ldots \in \mathbb{R}^2$ converges to a limit $(x, y)$ if and only if $x_n \to x$ and $y_n \to y$ separately, as sequences in $\mathbb{R}$ with the usual metric.

**Solution:** The statement is true in all three metrics.

First suppose that $x_n \to x$ and $y_n \to y$ separately. Let $\epsilon > 0$ be given. Choose an integer $N_1$ such that $|x_n - x| < \epsilon/2$ for all $n \geq N_1$, and an integer $N_2$ such that $|y_n - y| < \epsilon/2$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$, and suppose that $n \geq N$. In the taxicab metric, we have

$$d_1((x_n, y_n),\, (x, y)) = |x_n - x| + |y_n - y| < (\epsilon/2) + (\epsilon/2) = \epsilon.$$

In the Euclidean metric, we have

$$\begin{aligned} d_2((x_n, y_n),\, (x, y)) &= \sqrt{|x_n - x|^2 + |y_n - y|^2} \\ &< \sqrt{(\epsilon/2)^2 + (\epsilon/2)^2} = \epsilon \cdot \sqrt{2}/2 < \epsilon. \end{aligned}$$

In the square metric, we have

$$d_\infty((x_n, y_n),\,(x, y)) = \max\{|x_n - x|, |y_n - y|\} < \epsilon/2 < \epsilon.$$

Thus $(x_n, y_n) \to (x, y)$ in all three metrics.

Conversely, suppose that $(x_n, y_n) \to (x, y)$ in any of the three metrics. In the taxicab metric we have

$$|x_n - x| \le |x_n - x| + |y_n - y| = d_1((x_n, y_n),\,(x, y)).$$

In the Euclidean metric we have

$$|x_n - x| = \sqrt{(x_n - x)^2} \le \sqrt{|x_n - x|^2 + |y_n - y|^2} = d_2((x_n, y_n),\,(x, y)).$$

In the square metric we have

$$|x_n - x| \le \max\{|x_n - x|,\,|y_n - y|\} = d_\infty((x_n, y_n),\,(x, y)).$$

The right-hand sides go to zero as $n \to \infty$, so $|x_n - x| \to 0$ as well, so $x_n \to x$. Similarly $y_n \to y$.

1.3. Consider the following silly metric on $\mathbb{R}^2$:

$$d((x_1, y_1),\,(x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1 - y_2| + 1 & \text{if } x_1 \ne x_2. \end{cases}$$

(a) Prove that $d$ is a metric, that is, it satisfies the three axioms.

**Solution:** Clearly $d$ is symmetric, $d(p, p) = 0$, and $d(p, q) > 0$ if $p \ne q$. It remains to check the triangle inequality:

$$d(p, q) \le d(p, r) \ge d(r, q).$$

Write $p = (x_1, y_1)$, $r = (x_2, y_2)$, and $q = (x_3, y_3)$. If $x_1 = x_3$ then we have

$$d(p, r) + d(r, q) \ge |y_1 - y_2| + |y_2 - y_3| \ge |y_1 - y_3| = d(p, q).$$

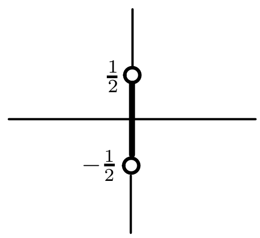
If $x_1 \ne x_3$ then either $x_1 \ne x_2$ or $x_2 \ne x_3$ or both. In any case we have

$$d(p, r) + d(r, q) \ge |y_1 - y_2| + |y_2 - y_3| + 1 \ge |y_1 - y_3| + 1 = d(p, q).$$

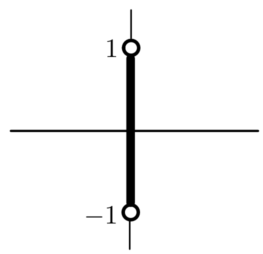
(b) Sketch the open balls of radius 1/2, 1, and 2 around the origin in this metric.

**Solution:**

1/2:




1:

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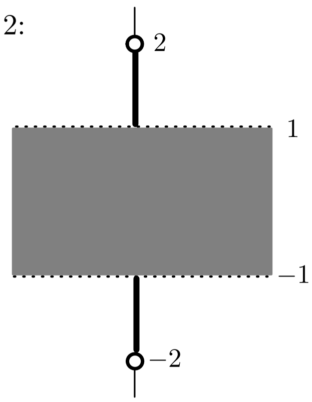


(c) Give an example of a sequence that converges in the Euclidean metric $d_2$ but not in our silly metric $d$.

**Solution:** Let $p_n = (\frac{1}{n}, 0)$. In the Euclidean metric we have $p_n \to (0,0)$ by problem 1.1(b) above. If it converged in the silly metric, say to a limit $\ell$, then there would be an $N$ such that $n \geq N$ implies $d(p_n, \ell) < 1/2$. In particular, we would have

$$d(p_N, p_{N+1}) \leq d(p_N, \ell) + d(\ell, p_{N+1}) < 1/2 + 1/2 = 1,$$

but in fact $d(p_m, p_n) = 1$ whenever $m \neq n$.

(In the terms of the next section, the sequence is not even Cauchy in the silly metric.)

(d) Show that every sequence that converges in $d$ converges $d_2$.

**Solution:** Let $p_n = (x_n, y_n)$ be a sequence converging to a limit $\ell = (x, y)$ in the silly metric $d$. We want to show that $p_n \to \ell$ in the Euclidean metric $d_2$.

Let $\epsilon > 0$ be given. Because $p_n \to \ell$ in $d$, there is an $N$ such that $d(p_n, \ell) < \min(\epsilon, 1)$ for all $n \geq N$. Because $d(p_n, \ell) < 1$, we must have $x_n = x$, so

$$d_2(p_n, \ell) = \sqrt{(x_n - x)^2 + (y_n - y)^2} = |y_n - y| = d(p_n, \ell) < \epsilon.$$

1.11. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, let $p_1, p_2, p_3, \dots$ be a sequence that converges to a point $\ell$ in $X$, and let $f \colon X \to Y$ be continuous at $\ell$. Prove that the sequence $f(p_1), f(p_2), f(p_3), \dots$ converges to $f(\ell)$ in $Y$.

**Solution:** Let $\epsilon > 0$ be given. Because $f$ is continuous at $\ell$, there is a $\delta > 0$ such that $d_Y(f(p_n), f(\ell)) < \epsilon$ whenever $d_X(p_n, \ell) < \delta$. Because $p_n$ converges to $\ell$, there is an integer $N$ such that $d_X(p_n, \ell) < \delta$ for all $n \geq N$. Thus $d_Y(f(p_n), f(\ell)) < \epsilon$ for all $n \geq N$.

1.4. (Optional.) In Example 1.8(a) we saw a sequence in $C([0,1])$ that converges in the $L^1$ metric but not in the sup metric. Prove that the reverse cannot happen: every sequence that converges in the sup metric converges in the $L^1$ metric.

**Solution:** First I claim that for any $f, g \in C([0,1])$ we have $d_1(f,g) \leq d_\infty(f,g)$. To see this, set

$$M = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Then $|f(x) - g(x)| \leq M$ for all $x \in [0,1]$, so

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx \leq \int_0^1 M\, dx = M = d_\infty(f,g).$$

Now suppose we have a sequence $f_1, f_2, f_3, \dots$ that converges in the sup metric to a limit $g$. Let $\epsilon > 0$ be given, and choose an $N$ such that $n \geq N$ implies $d_\infty(f_n, g) < \epsilon$; then $n \geq N$ also implies that $d_1(f_n, g) < \epsilon$. Thus the sequence converges to $g$ in the $L^1$ metric as well.