Solutions to Homework 3

2.6. Without using Proposition 2.12,

(a) Prove that if $U, V \subset X$ are open, then the intersection $U \cap V$ is again open.

Solution: Let $p \in U \cap V$. Then there is an $r_1 > 0$ such that $B_{r_1}(p) \subset U$, and an $r_2 > 0$ such that $B_{r_2}(p) \subset V$. Let $r = \min\{r_1, r_2\}$; then $B_r(p) \subset B_{r_1}(p) \subset U$, and $B_r(p) \subset B_{r_2}(p) \subset V$, so $B_r(p) \subset U \cap V$.

(b) Give an example of countably many open sets $U_1, U_2, U_3, \ldots \subset X$ such that their intersection $U_1 \cap U_2 \cap U_3 \cap \cdots$ is not open.

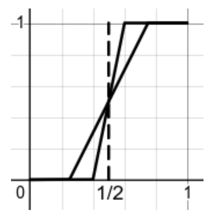
Solution: Let $X = \mathbb{R}$ with the usual metric, and let U_n be the open interval $(-\frac{1}{n}, \frac{1}{n})$. Then $U_1 \cap U_2 \cap \cdots = \{0\}$, which is not open.

(c) Let I be a set, and suppose that for each $i \in I$ we have an open set $U_i \subset X$. Prove that the union $\bigcup_{i \in I} U_i$ is again open.

Solution: Let $p \in \bigcup_{i \in I} U_i$. Then there is some $i \in I$ with $p \in U_i$, and because U_i is open, there is an $r > 0$ such that $B_r(p) \subset U_i$. Thus $B_r(p) \subset \bigcup_{i \in I} U_i$.

3.1. Prove that the sequence of piecewise-linear functions $f_2, f_3, f_4, \ldots \in C([0,1])$ introduced at the beginning of the section is Cauchy in the L^1 metric.

Solution: Recall that f_m was defined as going piecewise linearly from $f_n(0) = 0$ to $f_n(\frac{1}{2} - \frac{1}{n}) = 0$ to $f_n(\frac{1}{2} + \frac{1}{n}) = 1$ to $f_n(1) = 1$, so the graphs of f_m and f_n look like this:



The integral of $|f_m - f_n|$ is the area between the two graphs, that is, the area of two triangles whose height is $\frac{1}{2}$ and whose base is $|\frac{1}{m} - \frac{1}{n}|$, so

$$d_1(f_m, f_n) = \frac{1}{2}\left|\frac{1}{m} - \frac{1}{n}\right|.$$

So we must prove that for every $\epsilon > 0$ there is an integer N such that for all $m, n \geq N$ we have

$$\frac{1}{2}\left|\frac{1}{m} - \frac{1}{n}\right| < \epsilon.$$

If we're feeling sneaky, we could point out that this statement is equivalent to saying the sequence of real numbers $\frac{1}{2n}$ is Cauchy, and we know that $\frac{1}{2n} \to 0$, and that a convergent sequence is Cauchy, so we're done.

Or we could just prove the statement by hand. Let $\epsilon > 0$ be given, choose an N such that $\frac{1}{2N} < \epsilon$, and let $m, n \geq N$. If $n \geq m$ then $\frac{1}{m} \geq \frac{1}{n}$, so

$$\frac{1}{2}\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{2} \cdot \frac{1}{m} \leq \frac{1}{2} \cdot \frac{1}{N} < \epsilon,$$

and if $m \geq n$ then it's similar.

3.4. Let (X, d) be a metric space, let $f\colon X \to X$, suppose there is a "Lipschitz constant" $r \in [0, 1)$ such that for all $p, q \in X$ we have

$$d(f(p), f(q)) \leq r \cdot d(p, q).$$

Prove that f is continuous.

Solution: Let $p \in X$ and $\epsilon > 0$ be given, and set $\delta = \epsilon$. If $d(p, q) < \delta$, then

$$d(f(p), f(q)) \leq r \cdot d(p, q) < d(p, q) < \delta = \epsilon,$$

where in the second step we used the fact that $r < 1$.

3.3. (Optional.) Let $p_1, p_2, p_3, \ldots$ be a Cauchy sequence in a metric space (X, d). Prove that the sequence is bounded, meaning that there is a point $q \in X$ and a radius $R > 0$ such that $p_n \in B_R(q)$ for all n. In fact, for any $q \in X$ you can find such a radius R, and in particular for $X = \mathbb{R}$ you can take $q = 0$.

Solution: I'll prove the stronger statement, "for any q there is a radius $R\ldots$" rather than "there is a q and a radius $R\ldots$"

Let $q \in X$ be given. Apply the definition of Cauchy with $\epsilon = 1$ to get an N such that for all $m, n \geq N$ we have $d(p_m, p_n) < 1$, and in particular $d(p_N, p_n) < 1$. Let R be the maximum of $d(q, p_1) + 1$, $d(q, p_2) + 1$, and so on up through $d(q, p_N) + 1$.

I claim that $p_n \in B_R(q)$ for all n, that is, $d(q, p_n) < R$ for all n. If $n \leq N$ then this is clear by construction. If $n \geq N$ then $d(p_N, p_n) < 1$, so

$$d(q, p_n) \leq d(q, p_N) + d(p_N, p_n) < d(q, p_N) + 1 \leq R.$$