Solutions to Homework 4

2.8 In Example 1.4 we saw three different metrics on $\mathbb{R}^2$. Prove one of the following:

(a) A subset $A \subset \mathbb{R}^2$ is open in the Euclidean metric if and only if it is open in the taxicab metric.

(b) A subset $A \subset \mathbb{R}^2$ is open in the Euclidean metric if and only if it is open in the square metric.

(c) A subset $A \subset \mathbb{R}^2$ is open in the taxicab metric if and only if it is open in the square metric.

Solution: I'll prove all three. Given two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, I claim first that

$$d_\infty(\mathbf{x}, \mathbf{y}) \le d_2(\mathbf{x}, \mathbf{y}) \le d_1(\mathbf{x}, \mathbf{y}) \le 2d_\infty(\mathbf{x}, \mathbf{y}). \tag{1}$$

I'll prove it assuming that $|x_1 - y_1| \ge |x_2 - y_2|$; the other case is similar.

For the first inequality of (1), we have $d_\infty(\mathbf{x}, \mathbf{y}) = |x_1 - y_1|$, so

$$d_\infty(\mathbf{x}, \mathbf{y})^2 = |x_1 - y_1|^2 \le |x_1 - y_1|^2 + |x_2 - y_2|^2,$$

and taking square roots we get $d_\infty(\mathbf{x}, \mathbf{y}) \le d_2(\mathbf{x}, \mathbf{y})$.

For the second inequality of (1), use the triangle inequality in d_2:

$$\begin{aligned} d_2(\mathbf{x}, \mathbf{y}) &\le d_2(\mathbf{x}, (x_2, y_1)) + d_2((x_2, y_1), \mathbf{y}) \\ &= |x_1 - x_2| + |y_1 - y_2| = d_1(\mathbf{x}, \mathbf{y}). \end{aligned}$$

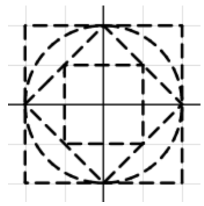
For the third inequality of (1), we have

$$d_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| \le |x_1 - y_1| + |x_1 - y_1| = 2d_\infty(\mathbf{x}, \mathbf{y}).$$

Thus (1) is established.

It follows that for any $r > 0$, the balls in the three metrics are related by

$$B_{r/2}(\mathbf{x}, d_\infty) \subset B_r(\mathbf{x}, d_1) \subset B_r(\mathbf{x}, d_2) \subset B_r(\mathbf{x}, d_\infty). \tag{2}$$

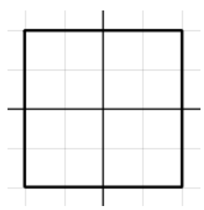


If A is open in d_∞, then for any $\mathbf{x} \in A$ there is an $r > 0$ such that $B_r(\mathbf{x}, d_\infty) \subset A$, so $B_r(\mathbf{x}, d_2) \subset A$ by (2); thus A is open in d_2. Similarly, if A is open in d_2 then it is open in d_1, and if A is open in d_1 then it is open in d_∞.

4.1 Find the closure, interior, and boundary of each subset of $\mathbb{R}^2$ in the Euclidean topology:

(a) $A_1 = \{(x, y) : 0 < x \leq 1, 0 \leq y < 1\}$

Solution: The interior is the open square given by $0 < x < 1$ and $0 < y < 1$. The closure is the closed square $0 \leq x \leq 1$ and $0 \leq y \leq 1$. The boundary is the four line segments shown:



(b) $A_2 = \{(x, y) : 0 < x \leq 1, y = 0\}$

Solution: The interior is empty. The closure is the line segment given by $0 \leq x \leq 1$ and $y = 0$. The boundary is the same line segment.

(c) $A_3 = \{(x, y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$

Solution: The interior is empty. The closure is all of $\mathbb{R}^2$. The boundary is all of $\mathbb{R}^2$.

(d) The subset A from Exercise 2.1.

The interior is the complement of the y-axis, that is, the set where $x \neq 0$. The closure is all of $\mathbb{R}^2$. The boundary is the y-axis.

4.2 Prove the analogue of Proposition 4.10 for closures without using Proposition 4.8. For (c), (d), and (e) especially, you'll want to follow the proof Proposition 4.10 closely.

(a) $\bar{A}$ is closed.

Solution: Let $p_1, p_2, \ldots$ be a sequence of points in $\bar{A}$ converging to a limit $\ell \in X$. We will prove that for every $\epsilon > 0$, the set $B_\epsilon(\ell) \cap A$ is not empty, so $\ell \in \bar{A}$ by Proposition 4.7.

Let $\epsilon > 0$ be given. Because $p_n \to \ell$, there is an N such that $d(p_n, \ell) < \epsilon/2$ for all $n \geq N$, and in particular $d(p_N, \ell) < \epsilon/2$. Because $p_N \in \bar{A}$, the set $B_{\epsilon/2}(p_N) \cap A$ is not empty by Proposition 4.7; let q be a point in that intersection. Then

$$d(q, \ell) \leq d(q, p_N) + d(p_N, \ell) < \epsilon/2 + \epsilon/2 = \epsilon,$$

so $B_\epsilon(\ell) \cap A$ contains the point q and is not empty.

(b) $\bar{A}$ is the smallest closed set contained in A, in the following sense: if $F \subset X$ is closed and $A \subset F$, then $\bar{A} \subset F$.

Solution: Let $F \subset X$ be a closed set with $A \subset F$; we will prove that $p \in \bar{A}$ implies $p \in F$. If $p \in \bar{A}$, then by definition there is a sequence $a_1, a_2, \ldots$ in A that converges to p. Because $A \subset F$, we have $a_n \in F$ for all n. Because F is closed, we have $p \in F$.

(c) If $A \subset B$ then $\bar{A} \subset \bar{B}$.

Solution: We have $A \subset B \subset \bar{B}$, and $\bar{B}$ is closed by part (a), so $\bar{A} \subset \bar{B}$ by part (b).

(d) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Solution: We have $A \subset \bar{A} \subset \bar{A} \cup \bar{B}$, and $B \subset \bar{B} \subset \bar{A} \cup \bar{B}$, so $A \cup B \subset \bar{A} \cup \bar{B}$. But $\bar{A} \cup \bar{B}$ is a union of two closed sets by part (a), hence is closed by Prop. 2.10, so $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ by part (b).

For the reverse inclusion, we have $A \subset A \cup B$, so $\bar{A} \subset \overline{A \cup B}$ by part (c), and similarly $\bar{B} \subset \overline{A \cup B}$; thus $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

(e) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. Give an example to show that the inclusion can be strict.

Solution: As in the first half of part (d), we have $A \cap B \subset A \subset \bar{A}$, and $A \cap B \subset B \subset \bar{B}$, so $A \cap B \subset \bar{A} \cap \bar{B}$. Now $\bar{A}$ and $\bar{B}$ are closed by part (a), and an intersection of closed sets is closed by Proposition 2.11, so $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$ by part (b).

For a counterexample to the reverse inclusion, take $X = \mathbb{R}$ in the usual metric, $A = (0, 1)$, and $B = (1, 2)$. Then $A \cap B = \varnothing$, so $\overline{A \cap B} = \varnothing$, but $\bar{A} = [0, 1]$ and $\bar{B} = [1, 2]$, so $\bar{A} \cap \bar{B} = \{1\}$.

4.3 Give an example to show that the inclusions $\overline{B_r(p)} \subset \bar{B}_r(p)$ and $\overline{B_r(p)} \subset \bar{B}_r(p)$ can be strict.

Solution: Let $X = \mathbb{Z}$ with the usual metric, let $p = 0$, and let $r = 1$. Then $B_r(p) = \{0\}$, which is both open and closed, whereas $\bar{B}_r(p) = \{-1, 0, 1\}$.

1.12 (Optional.) Let

$$W = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots, 0\}$$

with the metric induced from the usual one on $\mathbb{R}$. Let (X, d_X) be another metric space. Given a sequence $p_1, p_2, p_3 \ldots \in X$ and a point $\ell \in X$, prove that the map $f\colon W \to X$ defined by

$$\begin{cases} f(\frac{1}{n}) = p_n, \\ f(0) = \ell \end{cases}$$

is continuous if and only if $p_n \to \ell$.

Solution: First suppose that f is continuous. The sequence $\frac{1}{n}$ converges to 0 in W, so by Exercise 1.11 we see that the sequence $f(\frac{1}{n}) = p_n$ converges to $f(0) = \ell$.

Conversely, suppose that $p_n \to \ell$. We want to show that f is continuous at every $w \in W$. There are two cases: either $w = \frac{1}{n}$, or $w = 0$. Let $\epsilon > 0$ be given.

If $w = \frac{1}{n}$, let $\delta = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. If $w' \in W$ satisfies $d(w, w') < \delta$, then $w' = w$, so $d(f(w), f(w')) = 0 < \epsilon$.

If $w = 0$, choose N such that $d(p_n, \ell) < \epsilon$ for all $n > N$, and let $\delta = 1/N$. If $w' \in W$ satisfies $d(w, w') < \delta = 1/N$, then either $w' = 0$, so $d(f(w), f(w')) = 0 < \epsilon$, or $w' = 1/n$ for some $n > N$, so $d(f(w), f(w')) = d(\ell, p_n) < \epsilon$.