Solutions to Homework 5

5.2. Give examples to show that Theorem 5.4 can fail...

(a) If the subsets U_i are dense but not open.

Solution: Of course there are many possibilities for each of these, but here's one. Let $X = \mathbb{R}$ with the usual metric, let $U_1 = \mathbb{Q}$, and let $U_2 = R \setminus \mathbb{Q}$. Both are dense but not open, and $U_1 \cap U_2 = \varnothing$. (If you feel you need to to include $U_3, U_4, \ldots$, you can let them all be $\mathbb{Q}$, or $\mathbb{R} \setminus \mathbb{Q}$, or $\mathbb{R}$.)

(b) If the metric space X is not complete.

Solution: Let $X = \mathbb{Q}$ with the usual metric inherited from $\mathbb{R}$, enumerate all the points of $\mathbb{Q}$ as $x_1, x_2, x_3, \ldots$, and let $U_i = \mathbb{Q} \setminus \{x_i\}$. Each U_i is open and dense, but $U_1 \cap U_2 \cap U_3 \cap \cdots = \varnothing$.

(c) If the collection of open, dense subsets is uncountable.

Solution: Let $X = \mathbb{R}$ with the usual metric, and for each $x \in \mathbb{R}$ take $U_x = \mathbb{R} \setminus \{x\}$. Then each U_x is open and dense, but $\bigcap_{x \in \mathbb{R}} U_x = \varnothing$.

5.3. Let (X, d) be any metric space (possibly incomplete), and let $U, V \subset X$ be two open, dense subsets. Prove that $U \cap V$ is again dense.

Solution: Recall that a subset is dense if and only if it intersects every non-empty open subset $W \subset X$. So let W be given, and let us argue that $W \cap U \cap V$ is not empty. Because U is dense, W intersects U; let $p \in W \cap U$. Because W and U are open, $W \cap U$ is open, so there is an $r > 0$ such that $B_r(p) \subset W \cap U$. Because V is dense, $B_r(p)$ intersects V. Because $B_r(p) \subset W \cap U$ we see that $W \cap U \cap V$ is not empty, which is what we wanted.

6.1. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f(A)$ for the following subsets $A \subset \mathbb{R}$: the intervals $[-1,1]$, $[-1,1)$, $(-1,1)$, $[0,1]$, $[0,1)$, and $(0,1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

Solution:

$$\begin{array}{rrr} f([-1,1]) = [0,1] & f([-1,1)) = [0,1] & f((-1,1)) = [0,1) \\ f([0,1]) = [0,1] & f([0,1)) = [0,1] & f((0,1)) = [0,1) \\ f(\{-1\}) = \{1\} & f(\{0\}) = \{0\} & f(\{1\}) = \{1\}. \end{array}$$

(b) Now let $f\colon X \to Y$ be arbitrary, and let $A, B \subset X$. Prove that if $A \subset B$ then $f(A) \subset f(B)$. Prove that $f(A \cup B) = f(A) \cup f(B)$. Prove that $f(A \cap B) \subset f(A) \cap f(B)$, but give an example where they are not equal.

Solution: First we show that if $A \subset B$ then $f(A) \subset f(B)$. Let $y \in f(A)$, so there is an $a \in A$ such that $f(a) = y$. Because $A \subset B$, we have $a \in B$, so there is an $a \in B$ such that $f(a) = y$, so $y \in f(B)$.

Next we show that $f(A \cup B) = f(A) \cup f(B)$. First, we have $A \subset A \cup B$, so $f(A) \subset f(A \cup B)$, and similarly $f(B) \subset f(A \cup B)$, so $f(A) \cup f(B) \subset f(A \cup B)$. For the reverse inclusion, let $y \in f(A \cup B)$. Then there is an $x \in A \cup B$ such that $f(x) = y$. Either $x \in A$, in which case $y = f(x) \in f(A)$, or $x \in B$, in which case $y = f(x) \in f(B)$; in either case $y \in f(A) \cup f(B)$.

Next we show that $f(A \cap B) \subset f(A) \cap f(B)$. We have $A \cap B \subset A$, so $f(A \cap B) \subset f(A)$, and $A \cap B \subset B$, so $f(A \cap B) \subset f(B)$.

Finally we give a counterexample to $f(A \cap B) = f(A) \cap f(B)$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the map $f(x) = x^2$, let $A = \{-1\} \subset \mathbb{R}$, and let $B = \{1\} \subset \mathbb{R}$. Then $f(A) = \{1\}$ and $f(B) = \{1\}$, so $f(A) \cap f(B) = \{1\}$, but $f(A \cap B) = f(\varnothing) = \varnothing$.

6.2. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f^{-1}(B)$ for the following subsets $B \subset \mathbb{R}$: the intervals $[-1,1]$, $[-1,1)$, $(-1,1)$, $[0,1]$, $[0,1)$, and $(0,1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

Solution:

$$\begin{array}{ccc} f^{-1}([-1,1]) = [-1,1] & f^{-1}([-1,1)) = (-1,1) & f^{-1}((-1,1)) = (-1,1) \\ f^{-1}([0,1]) = [-1,1] & f^{-1}([0,1)) = (-1,1) & f^{-1}((0,1)) = (-1,0) \cup (0,1) \\ f^{-1}(\{-1\}) = \varnothing & f^{-1}(\{0\}) = \{0\} & f^{-1}(\{1\}) = \{-1,1\}. \end{array}$$

(b) Now let $f\colon X \to Y$ be arbitrary, and let $A, B \subset Y$. Prove that if $A \subset B$ then $f^{-1}(A) \subset f^{-1}(B)$. Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, and that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Solution: First we show that if $A \subset B$ then $f^{-1}(A) \subset f^{-1}(B)$. Let $x \in f^{-1}(A)$. Then $f(x) \in A$, so $f(X) \in B$, so $x \in f^{-1}(B)$.

Next we show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. First, we have $A \subset A \cup B$, so $f^{-1}(A) \subset f^{-1}(A \cup B)$, and similarly $f^{-1}(B) \subset f^{-1}(A \cup B)$, so $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$. For the reverse inclusion, let $x \in f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$; if $f(x) \in A$ then $x \in f^{-1}(A)$, and if $f(x) \in B$ then $x \in f^{-1}(B)$, and in either case $x \in f^{-1}(A) \cup f^{-1}(B)$. Thus $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$.

Finally we show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. First, we have $A \cap B \subset A$, so $f^{-1}(A \cap B) \subset f^{-1}(A)$, and $A \cap B \subset B$, so $f^{-1}(A \cap B) \subset f^{-1}(B)$; thus $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$. For the reverse inclusion, let $x \in f^{-1}(A) \cap f^{-1}(B)$; then $x \in f^{-1}(A)$, so $f(x) \in A$, and $x \in f^{-1}(B)$, so $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. Thus $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$ as desired.

5.7. Give examples to show that Proposition 5.6 can fail...

(c) If the diameters all finite, but do not go to zero.

Solution: Following the hint, take $X = C([0,1])$ with the sup metric, and let F_n be the set of continuous functions $f\colon [0,1] \to [0,1]$ with $f(0) = 1$ and $f(x) = 0$ for $x \geq \frac{1}{n}$. We can see that

$$F_1 \supset F_2 \supset \cdots .$$

First let us argue that each F_n is closed. Suppose that $f_1, f_2, \ldots$ is a sequence of functions in F_n converging to a limit $f \in C([0,1])$. By Example 1.8(c), evaluation at $x = 0$ is continuous in the sup metric, so by Exercise 1.11 it preserves limits of sequences, so we have

$$f(0) = \lim_{k\to\infty} f_k(0) = 1.$$

Similarly, for all $x \geq \frac{1}{n}$ we have

$$f_k(x) = \lim_{k\to\infty} f_k(0) = 0.$$

For $0 < x < \frac{1}{n}$ we have $0 \leq f_k(x) \leq 1$, and taking the limit as $k \to \infty$ we get $0 \leq f(x) \leq 1$. Thus $f \in F_n$, as desired.

Next let us argue that $\operatorname{diam}(F_n) = 1$. Given two functions $f, g \in F_n$, both functions take values in $[0,1]$, so $|f(x) - g(x)| \leq 1$ for all $x \in [0,1]$, so $d_\infty(f,g) \leq 1$. Thus

$$\operatorname{diam} F_n = \sup_{f,g\in F_n} d_\infty(f,g) \leq 1.$$

It remains to produce two functions $f, g \in F_n$ with $d_\infty(f,g) = 1$. Let f be the piecewise-linear function that goes from $f(0) = 1$ to $f(\frac{1}{2n}) = 1$ to $f(\frac{1}{n}) = 0$ to $f(1) = 0$, and let g be the piecewise-linear function that goes from $f(0) = 1$ to $f(\frac{1}{2n}) = 0$ to $f(1) = 0$; then both are in F_n, and $|f(\frac{1}{2n}) - g(\frac{1}{2n})| = |1 - 0| = 1$, so $d_\infty(f,g) = 1$.

Finally, let us argue that $F_1 \cap F_2 \cap \cdots$ is empty. Observe that if f were in the intersection then for all $n \geq 1$ and all $x \geq \frac{1}{n}$ we would have $f(x) = 0$, so for all $x > 0$ we would have $f(x) = 0$; on the other hand, $f(0) = 1$, so f would not be continuous.

---

*I see that I mistakenly wrote 5.6(c) on the assignment. Hopefully you figured out that problem 5.6 didn't have parts and I meant 5.7(c), but if you just did 5.6 that's ok.