Solutions to Homework 6

7.1. Prove that each of the four topologies on $\mathbb{R}$ given in Example 7.3 is a topology, that is, it satisfies the three conditions in Definition 7.1.

(a) The finite complement topology: $U \subset \mathbb{R}$ is open if either $\mathbb{R} \setminus U$ is finite, or $U = \varnothing$.

Solution: The empty set $\varnothing$ is open by definition. To see that $\mathbb{R}$ is open, note that $\mathbb{R} \setminus \mathbb{R} = \varnothing$ is finite.

Suppose that U and V are open. If $U = \varnothing$ or $V = \varnothing$ then $U \cap V = \varnothing$, which is open. Otherwise $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ are finite, so $\mathbb{R} \setminus (U \cap V) = (\mathbb{R} \setminus U) \cup (\mathbb{R} \setminus V)$ is a union of two finite sets, hence is finite, so again $U \cap V$ is open.

Suppose that $\{U_i : i \in I\}$ is a collection of open sets. If I is empty, or every U_i is empty, then $\bigcup_{i \in I} U_i = \varnothing$, which is open. Otherwise there is some $i_0 \in I$ for which $\mathbb{R} \setminus U_{i_0}$ is finite. Then $\mathbb{R} \setminus \bigcup_{i \in I} U_i \subset \mathbb{R} \setminus U_{i_0}$ is a subset of a finite set, hence is finite, so again $\bigcup_{i \in I} U_i$ is open.

(b) The particular point topology: $U \subset \mathbb{R}$ is open if either $0 \in U$, or $U = \varnothing$.

Solution: The empty set $\varnothing$ is open by definition. To see that $\mathbb{R}$ is open, note that $0 \in \mathbb{R}$.

We have $\varnothing \in T$ by definition, and $\mathbb{R} \in T$ because $0 \in \mathbb{R}$. .

Suppose that U and V are open. If $U = \varnothing$ or $V = \varnothing$ then $U \cap V = \varnothing$, which is open. Otherwise $0 \in U$ and $0 \in V$, so $0 \in U \cap V$, so again $U \cap V$ is open.

Suppose that $\{U_i : i \in I\}$ is a collection of open sets. If I is empty, or every U_i is empty, then $\bigcup_{i \in I} U_i = \varnothing$, which is open. Otherwise there is some $i_0 \in I$ for which $0 \in U_{i_0}$, so $0 \in \bigcup_{i \in I} U_i$, so again $\bigcup_{i \in I} U_i$ is open.

(c) The lower semi-continuous topology: the open sets $\varnothing$, $\mathbb{R}$, and intervals of the form (a, ∞).

Solution: The empty set $\varnothing$ and the whole set $\mathbb{R}$ are open by definition.

Suppose that U and V are open. Observe we either have $U \subset V$, in which case $U \cap V = U$, or $V \subset U$, in which case $U \cap V = V$, and in either case $U \cap V$ is open.

Suppose that $\{U_i : i \in I\}$ is a collection of open sets. If I is empty, or every U_i is empty, then $\bigcup_{i \in I} U_i = \varnothing$, which is open. If some $U_i = \mathbb{R}$, then $\bigcup_{i \in I} U_i = \mathbb{R}$, which is open. Otherwise we can discard the empty U_i's, shrinking the index set I if necessary, and write each U_i as (a_i, ∞) for some $a_i \in \mathbb{R}$. If the set of left endpoints $\{a_i : i \in I\}$ is unbounded below, then $\bigcup_{i \in I} U_i = \mathbb{R}$, which is open. If it is bounded below, let a be its infimum; then $\bigcup_{i \in I} U_i = (a, \infty)$, which is again open.

(d) The lower limit topology: a subset $U \subset \mathbb{R}$ is open if it can be written as a union of half-open intervals $[a, b)$.

Solution: The empty set can be written as the union of an empty collection of half-open intervals. The whole set $\mathbb{R}$ can be written as $\bigcup_{n \in \mathbb{Z}} [n, n+1)$.]
If $U = \bigcup_{i \in I} [a_i, b_i)$ and $V = \bigcup_{j \in J} [a_j, b_j)$, then

$$U \cap V = \bigcup_{i \in I,\, j \in J} [a_i, b_i) \cap [a_j, b_j).$$

But $[a_i, b_i) \cap [a_j, b_j) = [\max(a_i, a_j), \min(b_i, b_j))$, with the understanding that this empty $\max(a_i, a_j)$ is greater than $\min(b_i, b_j)$.

Let's not belabor $\bigcup_{i \in I} U_i$, but just say that a union of unions of $[a, b)$s is again a union of $[a, b)$s.

7.2. Find the interiors, closures, and boundaries of the following subsets $A \subset \mathbb{R}$ in the topologies from Example 7.3:

(a) $\mathbb{Z} \subset \mathbb{R}$ in the finite complement topology.

Solution: A set is open in this topology if and only if its complement is finite, or it is empty. Thus a set is closed if and only if it is finite, or is the whole space $\mathbb{R}$.

Now $\mathbb{Z}$ is infinite, so the only closed set containing $\mathbb{Z}$ is $\mathbb{R}$, so $\bar{\mathbb{Z}} = \mathbb{R}$. Similarly, $\mathbb{R} \setminus \mathbb{Z}$ is infinite, so $\overline{\mathbb{R} \setminus \mathbb{Z}} = \mathbb{R}$, so $\operatorname{int} \mathbb{Z} = \varnothing$, because the closure of the complement is the complement of the interior, as was true in metric spaces. Finally, $\partial \mathbb{Z} = \bar{\mathbb{Z}} \setminus \operatorname{int} \mathbb{Z} = \mathbb{R} \setminus \varnothing = \mathbb{R}$.

(b) $\{0\} \subset \mathbb{R}$ and $\{1\} \subset \mathbb{R}$ in the particular point topology.

Solution: A set is open in this topology if and only if it either contains 0, or is empty. Thus a set is closed if and only if it either does not contain 0, or is the whole space $\mathbb{R}$.

Thus $\{1\}$ is closed, and it contains no non-empty open set, so its interior is $\varnothing$, its closure is $\{1\}$, and its boundary is $\{1\}$, just as in the usual topology.

On the other hand, $\{0\}$ is open, and it is not contained in any closed set apart from the whole space $\mathbb{R}$, so its interior is $\{0\}$, its closure is $\mathbb{R}$, and its boundary is $(-\infty, 0) \cup (0, \infty)$, quite unlike the usual topology.

(c) $(0, 1) \subset \mathbb{R}$ in the lower semi-continuous topology.

Solution: The open sets in this topology are of the form (a, ∞) for $a \in \mathbb{R}$, together with the empty set and the whole set $\mathbb{R}$. Thus the closed sets are of the form $(-\infty, a]$ for $a \in \mathbb{R}$, together with the whole set $\mathbb{R}$ and the empty set.

A closed set containing $(0, 1)$ is either all of $\mathbb{R}$, or of the form $(-\infty, a]$ for some $a \geq 1$. The closure of $(0, 1)$ is the intersection of all these, which is $(-\infty, 1]$. The only open set contained in $(0, 1)$ is $\varnothing$, so its interior is $\varnothing$. Thus its boundary is $(-\infty, 1]$.

(d) $(0,1) \subset \mathbb{R}$ in the lower limit topology.

Solution: We see that the interior of a subset is the union of all half-open intervals $[a,b)$ that it contains.

Thus $(0,1)$ is its own interior, because we can write

$$(0,1) = [\tfrac{1}{2},1) \cup [\tfrac{1}{3},1) \cup [\tfrac{1}{4},1) \cup \cdots$$

To find the closure of $[0,1)$, let us find the interior of the complement $(-\infty,0] \cup [1,\infty)$. Taking the union of all half-open intervals $[a,b)$ contained in $(-\infty,0] \cup [1,\infty)$ gives $(-\infty,0) \cup [1,\infty)$. So the closure of $(0,1)$ is the complement of that, namely $[0,1)$. (It may seem strange that the closure is open, but it's true.)

The boundary of $(0,1)$ is $[0,1) \setminus (0,1) = \{0\}$.

8.1. Let X be a topological space. Let $Y \subset X$, and give Y the subspace topology. Let $A \subset Y$.

(a) Prove that if A is closed in Y and Y is closed in X, then A is closed in X.

Solution: Because A is closed in Y, by Proposition 8.5 there is a closed set $F \subset X$ such that $A = F \cap Y$. If Y is closed in X then this is an intersection of closed sets, hence is closed in X.

(b) Give two examples to show that if A is closed in Y and Y is *not* closed in X, then A may or may not be closed in X.

Solution: Let $X = \mathbb{R}$ with the usual topology, and let $Y = [0,3)$. If we take $A = [1,2]$ then A is closed X, so $A = A \cap Y$ is closed in Y. On the other hand, if we take $A = [2,3)$ then A is not closed in X, but $A = [2,3] \cap Y$, so A is closed in Y.

(c) Prove that if A is open in Y and Y is open in X, then A is open in X.

Solution: Because A is open in Y, there is an open set $U \subset X$ such that $A = U \cap Y$. If Y is open in X then this is an intersection of two open sets, hence is open in X.

(d) Give two examples to show that if A is open in Y and Y is *not* open in X, then A may or may not be open in X.

Solution: Let $X = \mathbb{R}$ with the usual topology, and let $Y = [0,3)$. If we take $A = (1,2)$ then A is open X, so $A = A \cap Y$ is open in Y. On the other hand, if we take $A = [0,1)$ then A is not open in X, but $A = (-1,1) \cap Y$, so A is open in Y.

(e) Let $A \subset Y$, let $\mathrm{cl}_X(A)$ denote the closure of A in X, and let $\mathrm{cl}_Y(A)$ denote the closure of A in Y. Prove that

$$\mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap Y.$$

Solution: First, $\mathrm{cl}_X(A) \cap Y$ is closed in Y and contains A, so it contains $\mathrm{cl}_Y(A)$.

For the reverse inclusion, we know that $\mathrm{cl}_Y(A)$ is closed in Y, so there is a closed $F \subset X$ such that $\mathrm{cl}_Y(A) = F \cap Y$. But $A \subset F$, so $\mathrm{cl}_X(A) \subset F$, so $\mathrm{cl}_X(A) \cap Y \subset F \cap Y = \mathrm{cl}_Y(A)$.

(f) Let $A \subset Y$, let $\mathrm{int}_X(A)$ denote the interior of A as a subset of X, and let $\mathrm{int}_Y(A)$ denote the interior of A as a subset of Y. Prove that

$$\mathrm{int}_X(A) \subset \mathrm{int}_Y(A).$$

Solution: We know that $\mathrm{int}_X(A)$ is open in X, so $\mathrm{int}_X(A) \cap Y$ is open in Y. But $\mathrm{int}_X(A) \subset A$ and $A \subset Y$, so $\mathrm{int}_X(A) \cap Y = \mathrm{int}_X(A)$. Thus $\mathrm{int}_X(A)$ is open in Y and is contained in A, hence is contained in $\mathrm{int}_Y(A)$.

(g) Give an example where the inclusion in part (f) is strict.

Solution: Let $X = \mathbb{R}$, let $Y = [0,2]$, and let $A = [0,1]$. Then $\mathrm{int}_X(A) = (0,1)$, but $\mathrm{int}_Y(A) = [0,1)$.

Or let $X = \mathbb{R}$, let $Y = \mathbb{Q}$, and let $A = (0,1) \cap \mathbb{Q}$. Then A is open in Y, so $\mathrm{int}_Y(A) = A$, but $\mathrm{int}_X(A) = \varnothing$.

8.2. (a) Let X be a topological space, and suppose that we can write $X = F_1 \cup \ldots \cup F_n$, where each F_i is closed. Let Y be another topological space, let $f\colon X \to Y$, and let $f_i\colon F_i \to Y$ be the restriction of f to F_i: that is, for $x \in F_i$ we set $f_i(x) = f(x)$.

Prove that f is continuous if and only if f_i is continuous for all i.

Solution: We freely use Proposition 7.7, which states that a map is continuous if and only if the preimage of every closed set is closed.

Let $G \subset Y$ be closed. Observe that

$$f_i^{-1}(G) = f^{-1}(G) \cap F_i.$$

If f is continuous then $f^{-1}(G)$ is closed in X, so $f^{-1}(G) \cap F_i$ is closed in F_i by definition of the subspace topology. Thus f_i is continuous.

Conversely, suppose that f_i is continuous for all i. Because $X = F_1 \cup \ldots \cup F_n$, we have

$$\begin{aligned} f^{-1}(G) &= f^{-1}(G) \cap (F_1 \cup \ldots \cup F_n) \\ &= (f^{-1}(G) \cap F_1) \cup \ldots \cup (f^{-1}(G) \cap F_n) \\ &= f_1^{-1}(G) \cup \ldots \cup f_n^{-1}(G). \end{aligned}$$

Because f_i is continuous, we know that $f_i^{-1}(G)$ is closed in F_i, hence is closed in X by Exercise 8.1(a). So $f^{-1}(G)$ is a finite union of closed sets, hence is closed. Thus f is continuous.

(b) This is usually applied to show that a piecewise function is continuous. Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1/3, \\ 3x - 1 & \text{if } 1/3 \leq x \leq 2/3, \\ 1 & \text{if } x \geq 2/3. \end{cases}$$

If we wanted to apply part (a) to show that f is continuous, which should sets should we take for the F_i?

Solution: Take $F_1 = [0, \frac{1}{3}]$, $F_2 = [\frac{1}{3}, \frac{2}{3}]$, and $F_3 = [\frac{2}{3}, 1]$. It is clear that the restriction of f to each of these is continuous, so f is continuous by part (a).

7.3. (Optional.) Let (X, d) be a metric space. A function $f: X \to \mathbb{R}$ is called *lower semi-continuous* at a point $p \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(p, q) < \delta$ implies $f(q) > f(p) - \epsilon$. The idea is that in the limit, f can only jump down. You can guess what *upper semi-continuous* means.

(a) Let $X = \mathbb{R}$ with the usual metric, and consider the floor function $\lfloor x \rfloor$, which returns the greatest integer $\leq x$, and the ceiling function $\lceil x \rceil$, which returns the least integer $\geq x$. Which one is lower semi-continuous, and which one is upper semi-continuous?

(You don't have to prove it.)

Solution: The ceiling function is lower semi-continuous; the floor function is upper semi-continuous.

If you wanted to prove it, you could start with the observation that if η is close to zero, then $\lfloor x + \eta \rfloor \leq \lfloor x \rfloor$, while $\lceil x + \eta \rceil \geq \lceil x \rceil$.

(b) Prove that $f: X \to \mathbb{R}$ is lower semi-continuous (at every point) if and only if it is continuous as a map of topological spaces when the codomain $\mathbb{R}$ is given the lower semi-continuous topology from Example 7.3(c).

Solution: First suppose that f is lower semi-continuous at every point. Take a subset $U \subset \mathbb{R}$ that's open the lower semi-continuous topology. If $U = \mathbb{R}$ then $f^{-1}(U) = X$, which is open, and if $U = \varnothing$ then $f^{-1}(U) = \varnothing$, which is open. If $U = (a, \infty)$ for some $a \in \mathbb{R}$ then $f^{-1}(U) = \{p \in X : f(p) > a\}$; I claim that this too open. Given a point $p \in f^{-1}(U)$, we have $f(p) > a$, so we can let $\epsilon = f(p) - a > 0$. Because f is lower semi-continuous, there is a $\delta > 0$ such that $d(p, q) < \delta$ implies $f(q) > f(p) - \epsilon = a$, which is equivalent to saying that $q \in B_\delta(p)$ implies $f(q) \in U$, or that $B_\delta(p) \subset f^{-1}(U)$. Thus $f^{-1}(U)$ is open, as desired.

Conversely, suppose that f is continuous when the codomain $\mathbb{R}$ is given the lower semi-continuous topology. Let $p \in X$ and $\epsilon > 0$ be given, and take $U = (f(p) - \epsilon, \infty) \subset \mathbb{R}$, which is open in the lower semi-continuous topology. Then $f^{-1}(U)$ is open in X, and we see that $p \in f^{-1}(U)$, so there is $\delta > 0$ such that $B_\delta(p) \subset f^{-1}(U)$. That is, if $d(p, q) < \delta$ then $q \in f^{-1}(U)$, which is true if and only if $f(q) \in U$, which is true if and only if $f(q) > f(p) - \epsilon$.

7.4. (Alternate optional problem.) Let (X, d) be a metric space. A function $f\colon \mathbb{R} \to X$ is called *continuous from the right* at a point $p \in \mathbb{R}$ if $\lim_{q\to p^+} f(q) = f(p)$: that is, for every $\epsilon > 0$ there is a $\delta > 0$ such that $p \leq q < p + \delta$ implies $d(f(p), f(q)) < \epsilon$. You can guess what *continuous from the left* means.

(a) Again take $X = \mathbb{R}$ in the usual metric, and consider the floor and ceiling functions. Which one is continuous from the right, and which one is continuous from the left?

(You don't have to prove it.)

Solution: The floor function is continuous from the right; the ceiling function is continuous from the left.

(b) Prove that a map $f\colon \mathbb{R} \to X$ is continuous from the right (at every point) if and only if it is continuous as a map of topological spaces when the domain $\mathbb{R}$ is given the lower limit topology from Example 7.3(d).

Solution: First suppose that f is continuous from the right at every point. We want to prove that for every open set $U \subset X$, the preimage $f^{-1}(U)$ is open in the lower limit topology. It is enough to prove that for every $p \in f^{-1}(U)$, there is a $\delta > 0$ such that the basic open set $[p, p+\delta) \subset f^{-1}(U)$. So let U and $p \in f^{-1}(U)$ be given. Then $f(p) \in U$, and because U is open, there is an $\epsilon > 0$ such that $B_\epsilon(f(p)) \subset U$. Because f is continuous from the right at p, there is a $\delta > 0$ such that $q \in [p, p+\delta)$ implies $f(q) \in B_\epsilon(f(p))$, which implies that $f(q) \in U$; thus $[p, p+\delta) \subset f^{-1}(U)$, as desired.

Conversely, suppose that f is continuous when the domain $\mathbb{R}$ is given the lower limit topology. Let $p \in X$ and $\epsilon > 0$ be given, and let U be the open ball $B_\epsilon(f(p)) \subset X$. Then $f^{-1}(U)$ is open in the lower limit topology, so it can be written as a union of intervals of the form $[a, b)$; we see that $p \in f^{-1}(U)$, so we have $p \in [a, b) \subset f^{-1}(U)$ for some $a < b$. Let $\delta = b - p$, which is positive. Then $[p, p+\delta) \subset [a, b) \subset f^{-1}(U)$, so if $q \in [p, p+\delta)$ then $f(q) \in B_\delta(f(p))$, as desired.