Solutions to Homework 7

9.7. Given a map $f\colon X \to Y$, we can consider its graph

$$\Gamma_f = \{(x, y) \in X \times Y : y = f(x)\}.$$

(a) Prove that if X and Y are topological spaces, Y is Hausdorff, and f is continuous, then Γ_f is closed.

Hint: You could do this by hand, or you could consider the preimage of the diagonal $\Delta \subset Y \times Y$ and under the map $X \times Y \to Y \times Y$ that sends (x, y) to $(f(x), y)$.

Solution: To do it by hand, let $W = (X \times Y) \setminus \Gamma_f$ and let us argue that W is open. Let $(x, y) \in W$, meaning that $f(x) \neq y$. Because Y is Hausdorff, there are disjoint open sets $U, V \subset Y$ with $f(x) \in U$ and $y \in V$. Because f is continuous, $f^{-1}(U)$ is open in X, and we see that $(x, y) \in f^{-1}(U) \times V$; it remains to prove that $f^{-1}(U) \times V \subset W$. Observe that if $f^{-1}(U) \times V$ intersected Γ_f, then there would be an $x' \in X$ with $(x', f(x')) \in f^{-1}(U) \times V$, which is to say that $x' \in f^{-1}(U)$ and $f(x') \in V$, so $f(x') \in U$ and $f(x') \in V$, which is impossible because $U \cap V = \varnothing$.

To do it the slick way, consider the projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$, which are continuous by Exercise 9.3 (which we did in lecture). To see that the map $X \times Y \to Y \times Y$ that sends (x, y) to $(f(x), y)$ is continuous, observe that its two componets are $f \circ p$ and q, which are both continuous, and apply Proposition 9.4. Because Y is Hausdorff, the diagonal

$$\Delta = \{(y_1, y_2) \in Y \times Y : y_1 = y_2\}$$

is closed in $Y \times Y$ by Proposition 9.6, so its preimage in $X \times Y$ is closed. But its preimage is the set of points $(x, y) \in X \times Y$ such that $(f(x), y) \in \Delta$, that is, $f(x) = y$, which is exactly Γ_f.

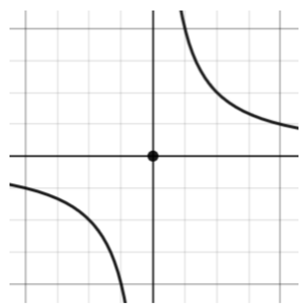
(b) Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that is not continuous (in the usual topology) but whose graph is nonetheless closed.

Hint: It won't work if f is bounded.

Solution: One possibility is

$$f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0. \end{cases}$$

The graph looks like this:



9.8. (a) Let X and Y be topological spaces, and suppose that Y is Hausdorff. Prove that if two continuous maps $f, g\colon X \to Y$ agree on a dense subset $D \subset X$, then $f = g$.

Hint: Let $E = \{x \in X : f(x) = g(x)\}$, and prove that it's closed.

Solution: The slick way is to consider the map $f \times g\colon X \to Y \times Y$, which sends a point x to $(f(x), g(x)$, and observe that $E = (f \times g)^{-1}(\Delta)$. As in my solution to Exercise 9.7, Δ is closed by Proposition 9.6, $f \times g$ is continuous by Proposition 9.4, and the preimage of a closed set is closed. Now E is closed and contains D, so E contains $\bar{D}$, but this is all of X because D is dense.

Or you can do it by hand as follows. Suppose there were some $x \in X$ with $f(x) \neq g(x)$. Because Y is Hausdorff, there are disjoint open sets $U, V \subset Y$ with $f(x) \in U$ and $g(x) \in V$. Thus $x \in f^{-1}(U) \cap g^{-1}(V)$, which is open because f and g are continuous. In fact we see that every point $x' \in f^{-1}(U) \cap g^{-1}(V)$ satisfies $f(x') \neq g(x')$, because $f(x') \in U$ and $g(x') \in V$, and $U \cap V = \varnothing$. Thus $f^{-1}(U) \cap g^{-1}(V)$ does not intersect D, but this is impossible because a dense set intersects every non-empty open set.

10.1. Endow $\mathbb{R}$ with the lower semi-continuous topology from Example 7.3(c).

(a) Prove that a subset $A \subset \mathbb{R}$ is compact in this topology if and only if it is bounded below and contains its infimum.

Solution: First suppose that $A \subset \mathbb{R}$ is bounded below, let $b = \inf A$, and suppose that $b \in A$. Let $\{U_i : i \in I\}$ be a covering of A by subsets of $\mathbb{R}$ that are open in the lower semi-continuous topology. Because $b \in A$, there is an $i_0 \in I$ such that $b \in U_{i_0}$, so either $U_{i_0} = \mathbb{R}$ or $U_{i_0} = (a, \infty)$ for some $a < b$. In either case $A \subset U_{i_0}$, so U_{i_0} by itself gives us a finite subcover. Thus A is compact.

Next let us argue that if A is compact, then A is bounded below. The open sets $U_1, U_2, \dots$ given by $U_m = (-m, \infty)$ form an open cover of A, because $U_1 \cup U_2 \cup \cdots = \mathbb{R}$. Because A is compact, we can extract a finite subcover $U_{m_1}, \dots, U_{m_k}$. Let $M = \max(m_1, \dots, m_k)$; then

$$U_{m_1} \cup \cdots \cup U_{m_k} = U_M = (-M, \infty),$$

and to say that this contains A is the same as saying that A is bounded below by $-M$.

Finally let us argue that if A is bounded below but does not contain $b = \inf A$, then A is not compact. The open sets $V_1, V_2, \dots$ given by $V_n = (b + \frac{1}{n}, \infty)$ form an open cover of A, because $V_1 \cup V_2 \cup \cdots = (b, \infty)$. Given any finite subcollection $V_{n_1}, \dots, V_{n_l}$, let $N = \max(n_1, \dots, n_l)$; then

$$V_{n_1} \cup \cdots \cup V_{n_l} = V_N = (b + \tfrac{1}{N}, \infty).$$

If this contained A, then $b + \frac{1}{N}$ would be a lower bound for A, but it is not, because b is the greatest lower bound. Thus our open cover has no finite subcover, so A is not compact.

(b) Let X be a compact space, and let $f\colon X \to \mathbb{R}$ be lower semi-continuous, that is, continuous with respect to the lower semi-continuous topology on $\mathbb{R}$. Prove that f achieves its minimum, that is, there is a point $p \in X$ such that $f(p) \leq f(x)$ for all $x \in X$.

Solution: We proved in lecture that the continuous image of a compact set is compact, so $f(X) \subset \mathbb{R}$ is compact in the lower semi-continuous topology, so $f(x)$ contains it infimum by part (a). Let $b = \inf f(X)$, and choose a point $p \in X$ such that $f(p) = b$. Then $f(p) \leq f(x)$ for all $x \in X$.

10.2. Endow $\mathbb{R}$ with the lower limit topology from Example 7.3(d). Prove that the subset $[0,1]$ is not compact.

Solution: Consider the open sets

$$[0, \tfrac{1}{2}), [\tfrac{1}{2}, \tfrac{2}{3}), [\tfrac{2}{3}, \tfrac{3}{4}), \ldots, [1, 2).$$

Their union is $[0,2)$ so they form an open cover of $[0,1]$. But we see that the union of any finite subcollection will not contain $[0,1]$, because it will miss $[\frac{n}{n+1}, 1)$ for some n.

10.4. (Optional.)

(a) Let X be a Hausdorff space. By definition, distint points of X have disjoint neighborhoods. In the proof of Proposition 10.6 we saw that a compact subset A and a point $p \notin A$ have disjoint neighborhoods. Prove that two disjoint compact sets $A, B \subset X$ have disjoint neighborhoods, that is, there are open sets $U, V \subset X$ with $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$.

Solution: We follow the second paragraph of the proof of Proposition 10.6. For every point $b \in B$, there are disjoint open sets $U_b \supset A$ and $V_b \ni b$, as it says in the statement of the problem. As b varies, the V_bs form an open cover of B, so we can extract a finite subcover: that is, we can choose $b_1, b_2, \ldots, b_n \in B$ such that $B \subset V_{b_1} \cup V_{b_2} \cup \cdots \cup V_{b_n}$. Let $U = U_{b_1} \cap \cdots \cap U_{b_n}$, and let $V = V_{b_1} \cup \cdots \cup V_{b_n}$; we see that U and V are disjoint, open, and we have $A \subset U$ and $B \subset V$, which is what we wanted.

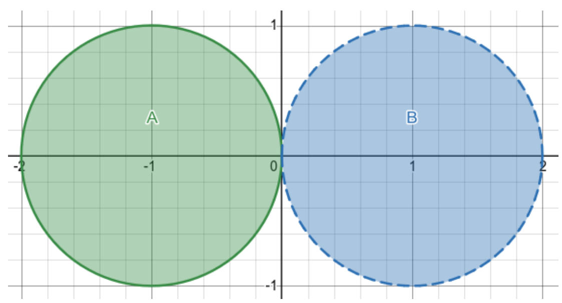
(b) Give an example of a non-Hausdorff space X, a compact subset $A \subset X$, and a point $p \in X \setminus A$ such that every neighborhood of p meets every neighborhood of A.

Solution: The cheap answer is to let X be any set with the indiscrete topology and at least two distinct points $p, q \in X$, and let $A = \{q\}$. Then A is compact (because it is finite), but p is only contained in one open set – the whole space X – and the same is true of A.

For a nicer example, let $X = \mathbb{R}$ with the lower semi-continuous topology, let $A = [1, 2)$, which is compact by Exercise 10.1(a), and let $p = 0$. An open set containing p is either $\mathbb{R}$ or (b, ∞) for some $b < 0$, and in either case it contains A, so it certainly meets any open set that contains A.

(c) Give an example of two subsets $A, B \subset \mathbb{R}^n$ that do not have disjoint neighborhoods. (Of course they cannot both be compact.)

Solution: In $\mathbb{R}^2$, let $A = \bar{B}_1((-1, 0))$ and $B = B_1((1, 0))$.



Then $(0, 0)$ is both in A and in the closure of B, so any open set that contains A must intersect B, and thus any open set that contains B.