Midterm 1

Friday, November 1, 2024

Use your own notebook paper, or get some from me. Feel free to do some scratch work that you don't turn in before writing the solution that you do turn in. I encourage you to write in pencil. There are 35 points in total.

1. (a) (3 points) Let (X, d) be a metric space. Define what it means for a sequence $p_1, p_2, \ldots$ in X to converge to a limit $\ell \in X$.

   (b) (5 points) Let $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ be two sequences in X, and suppose that $p_n \to \ell$ and $d(p_n, q_n) \to 0$ as $n \to \infty$. Prove that $q_n \to \ell$.

2. (a) (3 points) Let (X, d) be a metric space. Define what it means for a subset $A \subset X$ to be open.

   (b) (3 points) Write out what it means for a subset $A \subset X$ *not* to be open – that is, write out the negation of part (a).

   (c) (5 points) In $C([0,1])$, let A be the set of functions f with $f(0) > 0$. Prove that A is open in the sup metric.

   (d) (5 points) Prove that the same set $A \subset C([0,1])$ is not open in the L^1 metric.

3. (a) (3 points) Let (X, d) be a metric space, and let $A \subset X$. Define the interior, closure, and boundary of A.

   (b) (5 points) Prove that $\overline{A \setminus B} \subset \bar{A} \setminus \operatorname{int} B$. You may use anything proved in lecture or on the homework.

   (c) (3 points) Give an example to show that the inclusion in part (b) can be strict.