Solutions to Midterm 1

1. (a) (3 points) Let (X, d) be a metric space. Define what it means for a sequence $p_1, p_2, \ldots$ in X to converge to a limit $\ell \in X$.

   Solution: For every $\epsilon > 0$ there is a natural number N such that for all $n \geq N$ we have $d(p_n, \ell) < \epsilon$.

   (b) (5 points) Let $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ be two sequences in X, and suppose that $p_n \to \ell$ and $d(p_n, q_n) \to 0$ as $n \to \infty$. Prove that $q_n \to \ell$.

   Solution: Let $\epsilon > 0$ given. Because $p_n \to \ell$, we can choose an N_1 such that for all $n \geq N_1$ we have $d(p_n, \ell) < \epsilon/2$. Because $d(p_n, q_n) \to 0$, we can choose an N_2 such that for all $n \geq N_2$ we have $d(p_n, q_n) < \epsilon/2$. Let $N = \max\{N_1, N_2\}$; then for all $n \geq N$ we have

$$d(q_n, \ell) \leq d(q_n, p_n) + d(p_n, \ell) < \epsilon/2 + \epsilon/2 = \epsilon,$$

   where the first inequality is the triangle inequality.

2. (a) (3 points) Let (X, d) be a metric space. Define what it means for a subset $A \subset X$ to be open.

Solution: For every point $p \in A$ there is an $r > 0$ such that the ball $B_r(p) \subset A$, or if you prefer, for every $q \in X$ with $d(p, q) < r$ we have $q \in A$.

(b) (3 points) Write out what it means for a subset $A \subset X$ *not* to be open – that is, write out the negation of part (a).

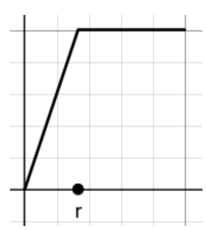
Solution: There is a point $p \in A$ such that for every $r > 0$ there is a point $q \in X$ with $d(p, q) < r$ but $q \notin A$.

(c) (5 points) In $C([0, 1])$, let A be the set of functions f with $f(0) > 0$. Prove that A is open in the sup metric.

Solution: Let $f \in A$ be given, and let $r = f(0) > 0$. If $g \in B_r(f)$ then $d_\infty(g, f) < r$, so $|g(x) - f(x)| < r$ for all $x \in [0, 1]$, an in particular $|g(0) - f(0)| < r$. Thus $f(0) - r < g(0) < f(0) + r$, that is, $0 < g(0) < 2f(0)$, so $g \in A$.

(d) (5 points) Prove that the same set $A \subset C([0, 1])$ is not open in the L^1 metric.

Solution: Let f be the constant function 1, which is in A, and let $r > 0$ be given. If $r > 1$, let g be the constant function 0; then $d_1(f, g) = \int |f - g| = 1 < r$, so $g \in B_r(f)$, but $g(0) = 0$ so $g \notin A$. More interestingly, if $r \leq 1$, let g be the function that goes piecewise-linearly from $g(0) = 0$ to $g(r) = 1$ to $g(1) = 1$:



Then $d_1(f, g) = \int |f - g|$ is the area of a triangle whose height is 1 and width is r, so the area is $r/2 < r$. So again $g \in B_r(f)$, but $g(0) = 0$ so $g \notin A$.

With a little more effort we could have proved that the interior of A is empty in this metric.

3. (a) (3 points) Let (X, d) be a metric space, and let $A \subset X$. Define the interior, closure, and boundary of A.

Solution: The interior is the set of points $p \in A$ for which there is an $r > 0$ such that $B_r(p) \subset A$. The closure is the set of points $p \in X$ for which there is a sequence $p_1, p_2, \ldots \in A$ converging to p. The boundary is the closure minus the interior.

(b) (5 points) Prove that $\overline{A \setminus B} \subset \bar{A} \setminus \operatorname{int} B$. You may use anything proved in lecture or on the homework.

Solution: Let $C = X \setminus B$, so $A \setminus B = A \cap C$. From homework we know that $\overline{A \cap C} \subset \bar{A} \cap \bar{C}$, and from lecture we know $\bar{C} = X \setminus \operatorname{int} B$, so $\bar{A} \cap \bar{C} = \bar{A} \setminus \operatorname{int} B$.

Alternatively you could say that $A \subset \bar{A}$ and $\operatorname{int} B \subset B$, so $\bar{A} \setminus \operatorname{int} B$ contains $A \setminus B$, and it's closed (because it's an intersection of two closed sets, $\bar{A}$ and $X \setminus \operatorname{int} B$), so it contains $\overline{A \setminus B}$.

(c) (3 points) Give an example to show that the inclusion in part (b) can be strict.

Solution: You might take $A = [0, 2]$ and $B = [0, 1]$ in $\mathbb{R}$ with the usual metric. Then $A \setminus B = (1, 2]$, whose closure is $[1, 2]$, but

$$\bar{A} \setminus \operatorname{int} B = [0, 2] \setminus (0, 1) = \{0\} \cup [1, 2],$$

which is bigger.

You could also take $A = B = \{0\}$, but this feels too smart-alecky.