4.2. The characteristic polynomial of \( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \) is \( \lambda^2 - 2a\lambda + a^2 - b^2 \), so the eigenvalues are \( a \pm b \), and in particular are always real. The analogue of the trace-determinant plane is as shown. Note the whole situation is symmetric under \( b \mapsto -b \), so we only sketch the upper half-plane.
4.6. We have seen in lecture that if $A, A'$ both have eigenvalues $\pm i\beta$, $\beta \neq 0$, then they are linearly conjugate, not just topologically conjugate. If you want to go through the proof, it is as follows. Let $V + iW$ be an $i\beta$-eigenvector for $A$. Then

$$A(V + iW) = i\beta(V + iW) = -\beta W + i\beta V,$$

so taking real and imaginary parts we find that

$$AV = -\beta W \quad \text{and} \quad AW = \beta V.$$ 

If we consider the matrix whose columns are $V$ and $W$, we can write this as

$$A \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} V & W \end{pmatrix}.$$ 

Writing $T = \begin{pmatrix} V & W \end{pmatrix}$, we get

$$T^{-1}AT = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$ 

Similarly, if $V' + iW'$ is an $i\beta$-eigenvector for $A'$ and $T' = \begin{pmatrix} V' & W' \end{pmatrix}$, then

$$T'^{-1}A'T' = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}.$$ 

Thus

$$T^{-1}AT = T'^{-1}A'T',$$

so

$$A' = T'T^{-1}ATT'^{-1} = T'T^{-1}A(T'T^{-1})^{-1},$$

so $A'$ is linearly conjugate to $A$.

Next suppose that $A$ has eigenvalues $\pm i\beta$, $\beta \neq 0$, and $B$ has eigenvalues $\pm i\gamma$, $\gamma \neq 0$. If $\gamma = -\beta$ then we are back in the previous situation, because $\pm i\gamma = \mp i\beta$. But if $|\gamma| \neq |\beta|$ then I claim that the two systems are not topologically conjugate. The point is that the period of a periodic solution is invariant under topological conjugacy.

Without loss of generality we can assume that $\beta > \gamma > 0$. By the argument above, there is a matrix $T$ such that

$$A = T \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} T^{-1},$$
so the flow of $X' = AX$ is

\[ X(t) = T \begin{pmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{pmatrix} T^{-1} X(0). \]

Thus $X(2\pi/\beta) = X(0)$, and if $X(0) \neq 0$ then there is no $t \in (0, 2\pi/\beta)$ with $X(t) = X(0)$. Similarly, for the flow of $Y' = BY$, we have $Y(2\pi/\gamma) = Y(0)$, and if $Y(0) \neq 0$ then there is no $t \in (0, 2\pi/\gamma)$ with $Y(t) = Y(0)$.

Now suppose that the systems $X' = AX$ and $Y' = BY$ are topologically conjugate. That is, there is a homeomorphism $h: \mathbb{R}^2 \to \mathbb{R}^2$ such that, for any solution $X(t)$ to the system $X' = AX$, the curve $Y(t) = h(X(t))$ is a solution to $Y' = BY$. Observe that $h(0) = 0$, because $h$ must take the equilibrium solution $X(t) \equiv 0$ of $X' = AX$ to an equilibrium solution of $Y' = BY$, and the only such solution is $Y(t) \equiv 0$. Now choose a solution $X(t)$ with $X(0) \neq 0$. We have $Y(0) = h(X(0)) \neq 0$, because $h$ is injective. Then

\[ Y(0) = h(X(0)) = h(X(2\pi/\beta)) = Y(2\pi/\beta), \]

but since $\beta > \gamma > 0$ we have $0 < 2\pi/\beta < 2\pi/\gamma$, so this contradicts what we said above, that there is no $t \in (0, 2\pi/\gamma)$ with $Y(t) = Y(0)$.

### 5.2.

(a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Characteristic polynomial: $-\lambda^3 + \lambda^2 + \lambda - 1 = -(\lambda - 1)^2(\lambda + 1)$

Eigenvalues: $1, 1, -1$

Eigenvalues:

\[ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \]

(b) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$

Characteristic polynomial: $-\lambda^3 + 2\lambda^2 + 3\lambda - 6 = -(\lambda - 2)(\lambda^2 - 3)$

Eigenvalues: $2, \pm \sqrt{3}$

Eigenvalues:

\[ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \pm \sqrt{3} \end{pmatrix} \]
(c) \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \)

Characteristic polynomial: \(-\lambda^3 + 3\lambda^2 = -\lambda^2(\lambda - 3)\)

Eigenvalues: 0, 0, 3

Eigenvectors: \( \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \)

(d) \( A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix} \)

Characteristic polynomial: \(-\lambda^3 + 2\lambda^2 - 4\lambda + 8 = -(\lambda - 2)(\lambda^2 + 4)\)

Eigenvalues: 2, \(\pm 2i\)

Eigenvectors: \( \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \pm i \end{pmatrix} \)

(e) \( A = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & -2 & -1 & -4 \\ -1 & 0 & 0 & 3 \end{pmatrix} \)

Characteristic polynomial:

\[ \lambda^4 - 6\lambda^3 + 13\lambda^2 - 18\lambda + 30 = (\lambda^2 + 3)(\lambda^2 - 6\lambda + 10) \]

Eigenvalues: \(\pm i\sqrt{3}, 3 \pm i\)

Eigenvectors: \( \begin{pmatrix} 0 \\ 2 \\ -1 \pm s \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \pm i \\ \mp i \end{pmatrix} \)

5.3. The characteristic polynomial of \( \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} \) is

\[ -\lambda^3 + b\lambda^2 + ac\lambda - abc = -(\lambda - b)(\lambda^2 - ac), \]

which has roots at \(b\) and \(\pm\sqrt{ac}\). If \(ac < 0\) then this is one real root and two purely imaginary roots. If \(ac > 0\) and \(b^2 \neq ac\) then this is three distinct real roots. If \(ac = 0\) and \(b \neq 0\), or if \(b^2 = ac > 0\), then this is a double real root and a distinct third real root. If \(ac = 0\) and \(b = 0\) then this is a triple root.
Last problem.

(a) I think it’s clear. One thing you might worry about is whether the map $\mathbb{R}^4 \to \mathbb{R}^4$ that sends a $2 \times 2$ matrix $M$ to $\exp(M)$ is continuous.

(b) First we check that $f(f^{-1}(X)) = X$ for all $X \in \mathbb{R}^2 \setminus \{0\}$:

$$f(f^{-1}(X)) = f(X/|X|, \log |X|)$$
$$= \exp(\log |X|) \cdot X/|X|$$
$$= |X| \cdot X/|X|$$
$$= X.$$ 

Next we check that $f^{-1}(f(U, t)) = (U, t)$ for all $U \in S^1$ and $t \in \mathbb{R}$. Note that

$$|\exp(t) U| = \exp(t) |U| = \exp(t),$$

so

$$f^{-1}(f(U, t)) = f^{-1}(\exp(t) U)$$
$$= (\exp(t) U/\exp(t), \log \exp(t))$$
$$= (U, t).$$

Thus $f$ and $f^{-1}$ are inverse to one another.

Next we check that $g(g^{-1}(X)) = X$ for all $X \in \mathbb{R}^2 \setminus \{0\}$:

$$g(g^{-1}(X)) = g(\exp(-A \log |X|/\alpha) X, \log |X|/\alpha)$$
$$= \exp(A \log |X|/\alpha) \exp(-A \log |X|/\alpha) X$$
$$= X.$$ 

In the last line, we need to know that $\exp(M) \exp(-M) = I$ for any matrix $M$. Either we just assert this, or we cite Prop. 3 from §6.4.

Last we check that $g^{-1}(g(U, t)) = (U, t)$ for all $U \in S^1$ and $t \in \mathbb{R}$. We know that

$$|\exp(At) U| = \exp(\alpha t) \cdot |U| = \exp(\alpha t),$$

so

$$\log |\exp(At) U| = \alpha t.$$ 

Thus we have

$$g^{-1}(g(U, t)) = g^{-1}(\exp(At) U)$$
$$= (\exp(-A \alpha t/\alpha) \exp(At) U, \alpha t/\alpha)$$
$$= (U, t),$$

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where again we have used \( \exp(-M) \exp(M) = I \). Thus \( g \) and \( g^{-1} \) are inverse to one another.

(c) It is clear that these are inverse to one another. If \( X \neq 0 \) then \( h(X) = g(f^{-1}(X)) \neq 0 \) and \( h^{-1}(X) = f(g^{-1}(X)) \neq 0 \), so
\[
\begin{align*}
    h(h^{-1}(X)) &= g(f^{-1}(f(g^{-1}(X)))) = g(g^{-1}(X)) = X \\
    h^{-1}(h(X)) &= f(g^{-1}(g(f^{-1}(X)))) = f(f^{-1}(X)) = X.
\end{align*}
\]
Moreover \( h(0) = 0 = h^{-1}(0) \), so
\[
\begin{align*}
    h(h^{-1}(0)) &= h(0) = 0 \\
    h^{-1}(h(0)) &= h^{-1}(0) = 0.
\end{align*}
\]
Thus \( h \) and \( h^{-1} \) are inverse to one another.

It is also clear that they are continuous away from zero, because \( f, f^{-1}, g, g^{-1} \) are continuous and \( h, h^{-1} \) are compositions of these.

For continuity at 0, we calculate:
\[
|h(X)| = |\exp(A \log |X|) X/|X| |
= \exp(\alpha \log |X|) |X|/|X|
= |X|^\alpha.
\]
\[
|h^{-1}(X)| = |\exp(\log |X|/\alpha) \exp(-A \log |X|/\alpha) X|
= \exp(\log |X|/\alpha) \exp(-\alpha \log |X|/\alpha) |X|
= |X|^{1/\alpha}.
\]
Since \( \alpha > 0 \) we have \( |X|^\alpha \to 0 \) and \( |X|^{1/\alpha} \to 0 \) as \( |X| \to 0 \).

(d) We have
\[
h(\exp(t)X_0) = g(f^{-1}(\exp(t) X_0))
= g(X_0/|X_0|, t + \log |X_0|)
= \exp(A(t + \log |X_0|)) g(X_0)/|X_0|,
\]
whereas
\[
\exp(At) h(X_0) = \exp(At) g(f^{-1}(X_0))
= \exp(At) g(X_0/|X_0|, \log |X_0|)
= \exp(At) \exp(A \log |X_0|) X_0/|X_0|
\]
which is the same. (Either we assert that

\[ \exp(At) \exp(As) = \exp(A(t+s)), \]

or we cite Prop. 2 from §6.4 for this fact.)

(e) I couldn’t see how to do this explicitly – that’s why I made it optional. My only approach would be to bash it over the head with the implicit function theorem. But I look forward to seeing if any of you found an explicit solution.