Solutions to Final Exam

1. Consider the linear system $X' = AX$, where

$$A = \begin{pmatrix} 3 & -5 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Find the eigenvalues of A.

Solution: It is an upper-triangular matrix, so we can read the eigenvalues off the diagonal: 3, -2, -1.

(b) Find the eigenvectors of A.

Solution:

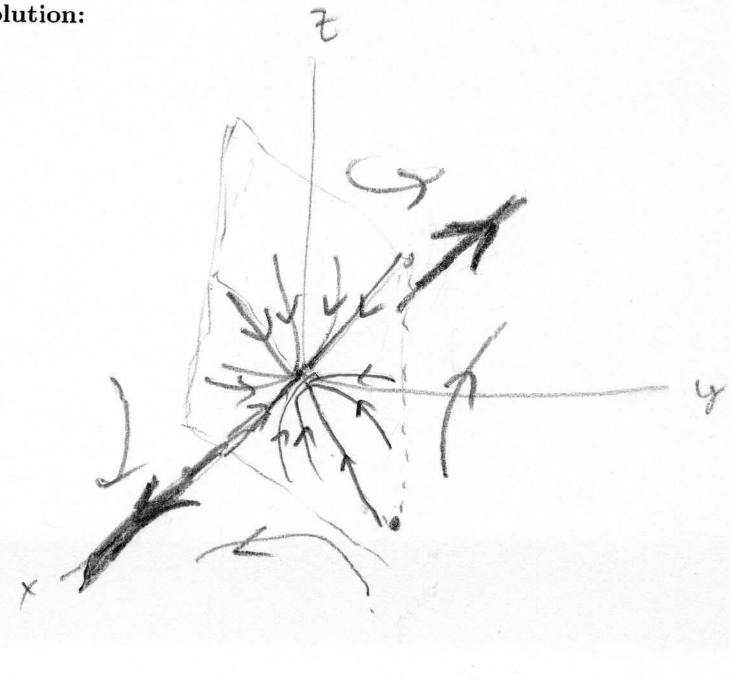
$$\lambda = 3 : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \lambda = -2 : \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda = -1 : \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) In $\mathbb{R}^3$, sketch the "stable subspace" spanned by the eigenvector(s) with negative eigenvalue(s). Sketch the flow of the linear system in this subspace *only*.

(d) On the same sketch, show the "unstable subspace" spanned by the eigenvector(s) with *positive* eigenvalue(s), and sketch the flow of the linear system in *this* subspace.

(e) Complete the sketch by adding a few flow lines outside of those two subspaces.

Solution:



2. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(a) Find $\exp(A)$ and $\exp(B)$.
Solution: The power series is

$$\exp(A) = I + A + \tfrac{1}{2}A^2 + \tfrac{1}{3!}A^3 + \cdots.$$

But $A^2 = 0 = B^2$, so

$$\exp(A) = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \exp(B) = I + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(b) Find $\exp(A+B)$.
Solution: One option is to observe that $(A+B)^2 = I$, so

$$\begin{aligned}
\exp(A+B) &= I + (A+B) + \tfrac{1}{2}I + \tfrac{1}{3!}(A+B) + \tfrac{1}{4!}I + \tfrac{1}{5!}(A+B) + \cdots \\
&= (1 + \tfrac{1}{2!} + \tfrac{1}{4!} + \cdots)I + (1 + \tfrac{1}{3!} + \tfrac{1}{5!} + \cdots)(A+B) \\
&= \cosh 1 \cdot I + \sinh 1 \cdot (A+B) \\
&= \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e + e^{-1} & e - e^{-1} \\ e - e^{-1} & e + e^{-1} \end{pmatrix}.
\end{aligned}$$

Alternatively, you could diagonalize $A+B$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1},$$

so $\exp(A+B)$ is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} e + e^{-1} & e - e^{-1} \\ e - e^{-1} & e + e^{-1} \end{pmatrix}.$$

(c) Find $\exp(A)\exp(B)$ and $\exp(B)\exp(A)$. Observe that these are different from one another, and from $\exp(A+B)$.
Solution:

$$\exp(A)\exp(B) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \exp(B)\exp(A) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

(d) State a sufficient condition for square matrices M and N to satisfy $\exp(M)\exp(N) = \exp(M+N) = \exp(N)\exp(M)$.
Solution: If $MN = NM$ then $\exp(M)\exp(N) = \exp(M+N) = \exp(N)\exp(M)$.

3. From reading and lecture: Consider the non-linear system

$$x' = \frac{1}{2}x - y - \frac{1}{2}x(x^2 + y^2)$$
$$y' = x + \frac{1}{2}y - \frac{1}{2}y(x^2 + y^2).$$

(a) Substitute $x = r\cos\theta$, $y = r\sin\theta$ into these equations.

Solution: The first becomes

$$r'\cos\theta - r\sin\theta \cdot \theta' = \frac{1}{2}r\cos\theta - r\sin\theta - \frac{1}{2}r^3\cos\theta.$$

The second becomes

$$r'\sin\theta + r\cos\theta \cdot \theta' = r\cos\theta + \frac{1}{2}r\sin\theta - \frac{1}{2}r^3\sin\theta.$$

(b) Take $\cos\theta$ times the first equation from (a), plus $\sin\theta$ times the second, and simplify to get a differential equation in r alone.

Solution:

$$r' = \frac{1}{2}r - \frac{1}{2}r^3.$$

(c) Take $\cos\theta$ times the *second* equation from (a), *minus* $\sin\theta$ times the *first*, and simplify to get a very simple differential equation in θ alone.

$$r\theta' = r.$$

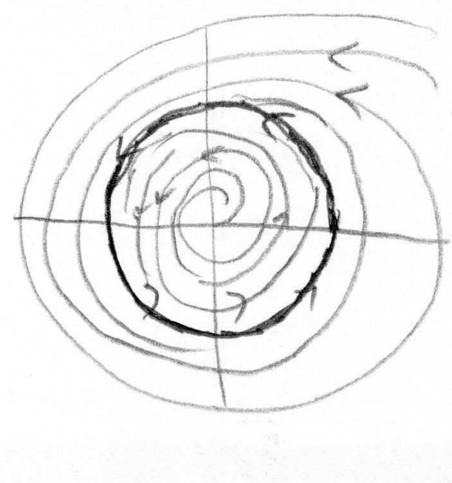
(d) Look at part (b). For which values of r is r increasing? Decreasing? Constant? Looking at part (c), what can you say about θ?

Solution: From part (b) we have $r' = \frac{1}{2}(r - r^3) = \frac{1}{2}r(1-r)(1+r)$, which is positive when $0 < r < 1$, negative when $r < 1$, and zero when $r = 0$ or $r = 1$. (Since we're working in polar coordinates, we're not interested in negative r.)

From part (c) we have $\theta' = 1$, so θ is increasing at constant speed: $\theta(t) = \theta(0) + t$.

(e) Sketch the flow of the system.

Solution:



4. Consider the non-linear system

$$x' = x - y^2$$
$$y' = y - x^2.$$

(a) There is an equilibrium point at the origin. What is the linearization there? Sketch the flow of the linearization.

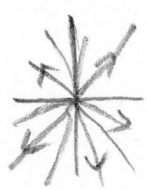
Solution: The linearization is

$$x' = x$$
$$y' = y,$$

or if you prefer a matrix,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The flow is a star-shaped source:



(b) There is one more equilibrium point. What is it?

Solution: We solve

$$x - y^2 = 0$$
$$y - x^2 = 0.$$

Thus $x = y^2$ and $y = x^2$, so $x = x^4$, so either $x = 0$, or $1 = x^3$ and thus $x = 1$. If $x = 0$ then $y = x^2 = 0$, so one equilibrium point is $(0, 0)$, which we already new about. If $x = 1$ then $y = x^2 = 1$, so the other equilibrium point is $(1, 1)$.

(c) What is the linearization at this second equilibrium point? What are the eigenvalues and eigenvectors? Sketch the flow of the linearization.

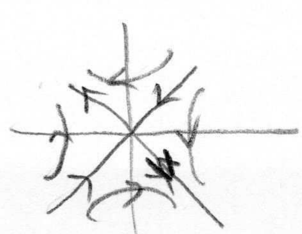
Solution: The Jacobian is

$$\begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -2y \\ -2x & 1 \end{pmatrix}.$$

Plugging in $x = 1$ and $y = 1$, we get

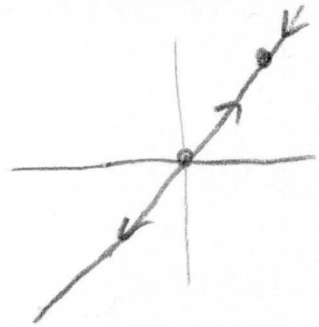
$$\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

The characteristic polynomial is $\lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3)$, so the eigenvalues are -1 and 3. The associated eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively. The phase portrait looks like this:



(d) Notice that the system is symmetric under reflecting across the diagonal line $y = x$. What does the flow of the non-linear system look like along that line?

Solution: If $y = x$ then $y' = x' = x - x^2 = x(1-x)$, which is positive for $0 < x < 1$, negative for $x < 0$ and $x > 1$, and zero for $x = 0$ or $x = 1$. So the flow looks like this:



(e) Just for fun, see which way the vector field is pointing at the four points $(\pm 10, \pm 10)$.

Solution:

$$\begin{aligned}
(10, 10) &: \quad x' = -90, \quad y' = -90 \\
(10, -10) &: \quad x' = -90, \quad y' = -110 \\
(-10, 10) &: \quad x' = -110, \quad y' = -90 \\
(-10, -10) &: \quad x' = -110, \quad y' = -110
\end{aligned}$$

In all cases the vector field points down and to the left at 45° or close to it.

(f) Sketch, very roughly, what the flow might look like.