Midterm
Math 420
April 29, 2016

Use your own notebook paper. Start each problem on a new sheet of paper.

1. Let \( A = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix} \).

(a) Find the eigenvalues of \( A \).

(b) Find eigenvectors for those eigenvalues.

(c) Find the general solution of \( X' = AX \).

(d) Sketch the phase portrait.

(e) If \( X(t) \) is a solution of \( X' = AX \) with \( X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), find \( X(1) \). What quadrant is it in? Does this agree with your picture?

2. Consider the second-order equation \( x'' + x' - 2x = 0 \).

(a) Convert this to a first-order system in two variables, find the eigenvalues and eigenvectors, and sketch the phase portrait.

(b) For which initial conditions, i.e. for which values of \( x(0) \) and \( x'(0) \), do we have \( x(t) \to \infty \) as \( t \to \infty ? \) For which do we have \( x(t) \to 0 \)? For which do we have \( x(t) \to -\infty \)?

3. Let \( A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda + \epsilon \end{pmatrix} \).

(a) Find the eigenvalues of \( A \).

(b) If \( \epsilon > 0 \), find eigenvectors for those eigenvalues.

(c) Let \( T \) be the matrix whose columns are those eigenvectors.

Find \( T^{-1} \). Check that \( T^{-1} T = I \) or \( TT^{-1} = I \).

(d) Check that \( T^{-1} AT = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + \epsilon \end{pmatrix} \).

(e) Show that if \( \epsilon = 0 \) then there is no \( T \) such that \( T^{-1} AT = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \).

Hint: What is \( T \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} T^{-1} \)?

(f) What happens to \( T^{-1} \) from part (d) as \( \epsilon \to 0 \)?