Solutions to Midterm

1. Let $A = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$.

   (a) Find the eigenvalues of A.
   Solution: The characteristic polynomial is

   $$\lambda^2 - 9\lambda + 18 = (\lambda - 3)(\lambda - 6),$$

   so the eigenvalues are 3 and 6.

   (b) Find eigenvectors for those eigenvalues.

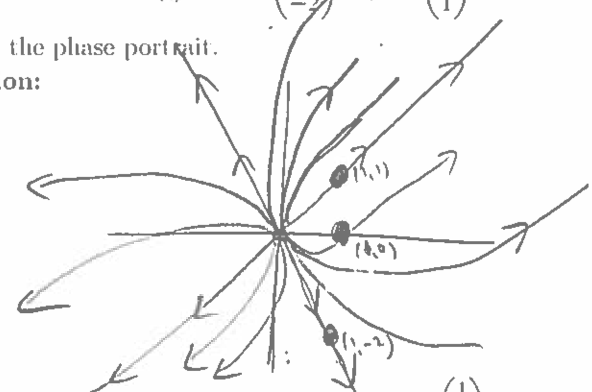
   Solution: For $\lambda = 3$ we can take any multiple of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

   For $\lambda = 6$ we can take any multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   (c) Find the general solution of $X' = AX$.
   Solution:

   $$X(t) = \alpha e^{3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \beta e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   (d) Sketch the phase portrait.
   Solution:



   (e) If $X(t)$ is a solution of $X' = AX$ with $X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find $X(1)$. What quadrant is it in? Does this agree with your picture?
   Solution: In the general solution from part (c) we need $\alpha = 1/3$ and $\beta = 2/3$, so

   $$X(1) = \begin{pmatrix} \frac{1}{3}e^3 + \frac{2}{3}e^6 \\ -\frac{2}{3}e^3 + \frac{2}{3}e^6 \end{pmatrix} \approx \begin{pmatrix} 275.6 \\ 255.6 \end{pmatrix}.$$

   This is in the first quadrant, which agrees with my picture.

2. Consider the second-order equation $x'' + x' - 2x = 0$.

(a) Convert this to a first-order system in two variables, find the eigenvalues and eigenvectors, and sketch the phase portrait.

Solution: We set $y = x'$, so

$$\begin{aligned} x' &= 0x + 1y \\ y' &= 2x - 1y, \end{aligned}$$

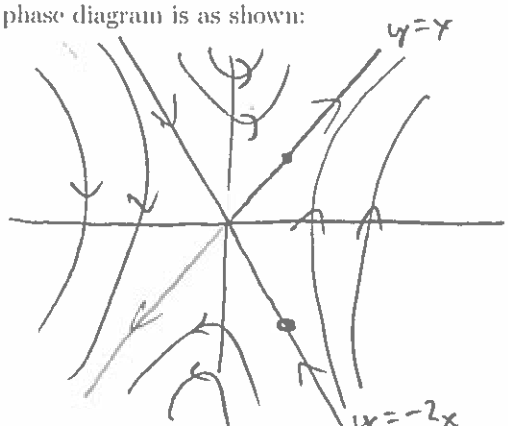
so

$$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}.$$

The characteristic polynomial is

$$\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2),$$

so the eigenvalues are 1 and -2. For the 1-eigenvector we can take any multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For the (-2)-eigenvector we can take any multiple of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. The phase diagram is as shown:



(b) For which initial conditions, i.e. for which values of $x(0)$ and $x'(0)$, do we have $x(t) \to \infty$ as $t \to \infty$? For which do we have $x(t) \to 0$? For which do we have $x(t) \to -\infty$?

Solution: From the picture we see that if we start with $y > -2x$ then we approach the line $y = x$ and head off to ∞. That is, if $x'(0) > -2x(0)$ then $x(t) \to \infty$ as $t \to \infty$. Similmarly, if $x'(0) < -2x(0)$ then $x(t) \to -\infty$. If $x'(0) = -2x(0)$ then $x(t) \to 0$; concretely, the solution is

$$x(t) = e^{-2t}x(0).$$

3. Let $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda + \epsilon \end{pmatrix}$.

(a) Find the eigenvalues of A.

Solution: Since A is upper-triangular, we can read the eigenvalues off the diagonal: λ and $\lambda + \epsilon$.

(b) If $\epsilon > 0$, find eigenvectors for those eigenvalues.

Solution: For the λ-eigenvector we can take $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For the $(\lambda + \epsilon)$-eigenvector we can take $\begin{pmatrix} 1 \\ \epsilon \end{pmatrix}$.

(c) Let T be the matrix whose columns are those eigenvectors. Find T^{-1}. Check that $T^{-1}T = I$ or $TT^{-1} = I$.

Solution: We have $T = \begin{pmatrix} 1 & 1 \\ 0 & \epsilon \end{pmatrix}$, so $T^{-1} = \begin{pmatrix} 1 & -1/\epsilon \\ 0 & 1/\epsilon \end{pmatrix}$. We check:

$$TT^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} 1 & -1/\epsilon \\ 0 & 1/\epsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$T^{-1}T = \begin{pmatrix} 1 & -1/\epsilon \\ 0 & 1/\epsilon \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & \epsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(d) Check that $T^{-1}AT = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + \epsilon \end{pmatrix}$.

Solution:

$$\begin{aligned} &\begin{pmatrix} 1 & -1/\epsilon \\ 0 & 1/\epsilon \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda + \epsilon \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & \epsilon \end{pmatrix} \\ &= \begin{pmatrix} \lambda & -\lambda/\epsilon \\ 0 & \lambda/\epsilon + 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & \epsilon \end{pmatrix} \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda + \epsilon \end{pmatrix} \end{aligned}$$

(e) Show that if $\epsilon = 0$ then there is no T such that $T^{-1}AT = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

Hint: What is $T \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} T^{-1}$?

Solution: If $T^{-1}AT = \lambda I$ then $A = T \cdot \lambda I \cdot T^{-1} = \lambda T \cdot T^{-1} = \lambda I$. But $A \neq \lambda I$, so there can be no such T.

(f) What happens to T^{-1} from part (d) as $\epsilon \to 0$?

Solution: The second column blows up.

It is also interesting to note that $\det T = \epsilon \to 0$, so T stops being invertible.