Problem for Homework 4

Math 420

Due April 27, 2016

Let us formalize the argument we sketched in lecture, showing that a spiral source $X' = AX$, where

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

with $\alpha > 0$ and $\beta \neq 0$, is topologically conjugate to a star-shaped source $X' = X$.

In lecture we calculated that

$$|\exp(At)\, V| = \exp(\alpha t)\ |V|$$

for $t \in \mathbb{R}$ and $V \in \mathbb{R}^2$.

(a) Let

$$S^1 = \left\{U \in \mathbb{R}^2 : |U| = 1\right\}$$

be the unit circle, and consider the following maps:

$$f\colon S^1 \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$$
$$f(U,t) = \exp(t)\, U$$

$$f^{-1}\colon \mathbb{R}^2 \setminus \{0\} \to S^1 \times \mathbb{R}$$
$$f^{-1}(X) = (X/\left|X\right|,\, \log\left|X\right|)$$

$$g\colon S^1 \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$$
$$g(U,t) = \exp(At)\, U,$$

$$g^{-1}\colon \mathbb{R}^2 \setminus \{0\} \to S^1 \times \mathbb{R}$$
$$g^{-1}(X) = \left(\exp\left(-A\log\left|X\right|/\alpha\right) X,\, \log\left|X\right|/\alpha\right).$$

Is it clear that these maps are continuous? If not, argue that they are.

(b) Show that f and f^{-1} are inverse to one another and that g and g^{-1} are inverse to one another, that is,

$$f(f^{-1}(X)) = X \qquad\qquad f^{-1}(f(U,t)) = (U,t)$$
$$g(g^{-1}(X)) = X \qquad\qquad g^{-1}(g(U,t)) = (U,t)$$

for all $X \in \mathbb{R}^2 \setminus \{0\}$ and $U \in S^1$ and $t \in \mathbb{R}$.

(c) Consider the maps $h, h^{-1} \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$h(X) = \begin{cases} 0 & \text{if } X = 0 \\ g(f^{-1}(X)) & \text{otherwise} \end{cases}$$

$$h^{-1}(X) = \begin{cases} 0 & \text{if } X = 0 \\ f(g^{-1}(X)) & \text{otherwise.} \end{cases}$$

Is it clear that these are inverse to one another, and continuous away from 0? If not, argue that they are. Then show that h and h^{-1} are continuous at 0, that is,

$$\lim_{|X| \to 0} |h(X)| = 0 \qquad\qquad \lim_{|X| \to 0} \left|h^{-1}(X)\right| = 0.$$

Thus h and h^{-1} are homeomorphisms.

(d) Show that h is a topological conjugacy from $X' = X$ to $X' = AX$, that is,

$$h\left(\exp(t)\, X_0\right) = \exp(At)\, h(X_0)$$

for all $X_0 \in \mathbb{R}^2$ and $t \in \mathbb{R}$.

(e) Optional. Maybe not feasible. If we replace A with the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},$$

how should we redefine g^{-1}?