Final Exam

Wednesday, December 5, 2018

Each part is worth 5 points, for a total of 65 points. The exam is long; feel free to do the parts you like best first. You may quote the result of an earlier problem or part even if you didn't manage to solve it.

1. (a) Define what it means for a topological space X to be connected.

   (b) Show that X is disconnected if and only if there is a continuous surjection $f\colon X \to Z$, where $Z = \{1, 2\}$ with the discrete topology. (Recall that the discrete topology means every subset is open.)

   (c) Use part (b) to show that if Y is a topological space and $A, B \subset Y$ are non-empty, connected subspaces with a common point $y \in A \cap B$, then $A \cup B$ is connected.

   (This was the key step in Homework 9 #1(a).)

2. Let X and Y be topological spaces, and let $p\colon X \times Y \to X$ be the projection $p(x, y) = x$.

   We have seen in lecture that p is continuous, and that p is an open map: that is, if $W \subset X \times Y$ is open then $p(W) \subset X$ is open.

   (a) Give an example to show that p need not be a closed map: that is, if $F \subset X \times Y$ is closed then $p(F) \subset X$ need not be closed.

   Hint: We've seen this in lecture; think about a hyperbola.

   (b) Define what it means for Y to be compact.

   (c) Prove the following lemma that we proved in lecture: If Y is compact, $W \subset X \times Y$ is open, and $\{x\} \times Y \subset W$ for some point $x \in X$, then there is an open set $U \subset X$ with $x \in U$ and $U \times Y \subset W$.

   Hint: Maybe start by drawing a picture to help you keep the notation straight.

(d) Show that if Y is compact then p is a closed map: that is, if $F \subset X \times Y$ is closed then $p(F) \subset X$ is closed.

Hint: Show that $X \setminus p(F)$ is open by showing that if $x \in X \setminus p(F)$ then there is an open $U \subset X$ such that $x \in U \subset X \setminus p(F)$. Apply part (c) with $W = (X \times Y) \setminus F$.

3. Let X and Y be topological spaces and let $f \colon X \to Y$ be any map. We define the *graph* of f,

$$\Gamma = \{(x, y) \in X \times Y : y = f(x)\}.$$

(a) Define what it means for Y to be Hausdorff. State (but do not prove) the characterization of Hausdorff in terms of the diagonal $\Delta \subset Y \times Y$.

(b) Show that if Y is Hausdorff and f is continuous then Γ is closed.

Hint: Consider the continuous map $\varphi \colon X \times Y \to Y \times Y$ given by $(x, y) \mapsto (f(x), y)$, and the diagonal $\Delta \subset Y \times Y$.

(c) Consider the projections

$$\begin{array}{ccc} X \times Y & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \\ X & & \end{array}$$

given by $p(x, y) = x$ and $q(x, y) = y$.

Show that for any subset $B \subset Y$ we have

$$f^{-1}(B) = p(\Gamma \cap q^{-1}(B)).$$

(d) Show that if Y is compact and Γ is closed, then f is continuous.

Hint: Show that the preimage of a closed set $G \subset Y$ is closed. Apply 3(c) and 2(d).

(e) Give an example to show that if Y is Hausdorff but not compact and Γ is closed then f need not be continuous.

Hint: f should "blow up" somewhere.

(f) Give an example to show that if Y is compact but not Hausdorff and f is continuous then Γ need not be closed.

Hint: The identity map is easiest.