Homework 2

Due Wednesday, October 3, 2018

1. Let X, Y, and Z be metric spaces, with metrics d_X, d_Y, and d_Z. Let $f\colon X \to Y$ be continuous at a point $p \in X$, and let $g\colon Y \to Z$ be continuous at $f(p)$. Show that $g \circ f$ is continuous at p.

2. Let (X, d) be a metric space, and fix a point $a \in X$. Show that the function $f\colon X \to \mathbb{R}$ given by $f(p) = d(p, a)$ is continuous.

   (Hint: Use the triangle inequality.)

3. Let X be any set, and let d_X be the *discrete metric*

$$d_X(p,q) = \begin{cases} 0 & \text{if } p = q, \text{ or} \\ 1 & \text{if } p \neq q. \end{cases}$$

   (a) Show that this is a metric.

   (b) Let (Y, d_Y) be another metric space (not necessarily discrete). Show that every map $f\colon X \to Y$ is continuous.

4. Consider $\mathbb{R}^2$ with either the Euclidean, taxicab, or square metric (your choice). Prove or give a counterexample to the following statement: a sequence of points $p_n = (x_n, y_n)$ converges to a limit $p = (x, y)$ if and only if $x_n \to x$ and $y_n \to y$ separately, as sequences in $\mathbb{R}$ with the usual metric.

5. Let p_n and q_n be two sequences in a metric space (X, d), and let $\ell \in X$. Suppose that $p_n \to \ell$ as $n \to \infty$, and $d(p_n, q_n) \to 0$ as $n \to \infty$. Show that $q_n \to \ell$ as $n \to \infty$.

6. **Optional:** Let $X = \mathbb{Q}$, let p be a prime number, and let d_p be the p-adic metric, defined as follows. Given $x, y \in \mathbb{Q}$, if $x \neq y$ then there is a unique integer n such that

$$x - y = p^n \cdot \frac{a}{b},$$

where a and b are integers not divisible by p, and we define $d_p(x, y) = p^{-n}$. If $x = y$ then we define $d_p(x, y) = 0$.

   (a) Write down a few rational numbers and find the 2-adic distance between them.

   (b) Show that the p-adic metric is a metric (for any prime p).

   (c) Show that the sequence $1, 3, 7, 15, 31, 63, 127, \ldots$ is Cauchy in the 2-adic metric, but does not converge to a limit. (Thus $\mathbb{Q}$ is not complete in the 2-adic metric.)

   Update: This is wrong: the sequence is Cauchy but it converges to -1. Here's an alternative problem, which is pretty hard but at least it's true. Consider the power series

$$(1+x)^{1/3} = 1 - \tfrac{1}{3}x + \tfrac{1\cdot 4}{3\cdot 3}x^2 - \tfrac{1\cdot 4\cdot 7}{3\cdot 3\cdot 3}x^3 + \tfrac{1\cdot 4\cdot 7\cdot 10}{3\cdot 3\cdot 3\cdot 3}x^4 - \cdots$$

   which converges absolutely (in the usual metric) for $x \in [-1, 1]$. Plugging in $x = 1$ we get

$$\sqrt[3]{2} = 1 - \tfrac{1}{3} + \tfrac{1\cdot 4}{3\cdot 3} - \tfrac{1\cdot 4\cdot 7}{3\cdot 3\cdot 3} + \tfrac{1\cdot 4\cdot 7\cdot 10}{3\cdot 3\cdot 3\cdot 3} - \cdots.$$

   You can show that this sequence of partial sums is Cauchy in the 2-adic metric because the terms have higher and higher powers of 2 in them. But it doesn't converge in $\mathbb{Q}$.

7. What is one question you have about last week's lectures?