Homework 3

Due Wednesday, October 10, 2018

1. Let $X = \mathbb{Q}$ with the usual metric. Show that the set
$$A = \{x \in \mathbb{Q} : x^2 < 1\}$$
is open but not closed, but that the set
$$B = \{x \in \mathbb{Q} : x^2 < 2\}$$
is both open and closed. (You can use the fact that $\sqrt{2}$ is irrational without proving it.)

2. Let $f\colon X \to Y$ be an arbitrary map of sets. The *image* of a subset $A \subset X$ is
$$f(A) = \{f(a) : a \in A\},$$
or equivalently
$$f(A) = \{y \in Y : y = f(a) \text{ for some } a \in A\}.$$
This is a subset of Y.

    (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f(A)$ for the following subsets $A \subset \mathbb{R}$: the intervals $[-1, 1]$, $[-1, 1)$, $(-1, 1)$, $[0, 1]$, $[0, 1)$, and $(0, 1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

    (b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x$. Choose a few subsets $A \subset \mathbb{R}^2$ and sketch both A and $f(A)$.

    (c) Now let $f\colon X \to Y$ be arbitrary, and let $A, B \subset X$. Show that if $A \subset B$ then $f(A) \subset f(B)$. Show that $f(A \cup B) = f(A) \cup f(B)$. Show that $f(A \cap B) \subset f(A) \cap f(B)$, but give an example where the two are not equal.

3. Let $f\colon X \to Y$ be an arbitrary map of sets. The *preimage* of a subset $B \subset Y$ is

$$f^{-1}(B) = \{x \in X : f(x) \in B\},$$

or equivalently

$$f^{-1}(B) = \{x \in X : f(x) = b \text{ for some } b \in B\}.$$

This is a subset of X.

   (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f^{-1}(B)$ for the following subsets $B \subset \mathbb{R}$: the intervals $[-1, 1]$, $[-1, 1)$, $(-1, 1)$, $[0, 1]$, $[0, 1)$, and $(0, 1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

   (b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = xy$. Sketch $f^{-1}(B)$ when B is $\{0\}$, $\{1\}$, $[0, 1]$, $(0, 1)$, and $[0, 1)$.

   (c) Now let $f\colon X \to Y$ be arbitrary, and let $A, B \subset Y$. Show that if $A \subset B$ then $f^{-1}(A) \subset f^{-1}(B)$. Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, and that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

4. Let $f\colon X \to Y$, let $A \subset X$, and let $B \subset Y$.

   (a) Show that $A \subset f^{-1}(B)$ if and only if $f(A) \subset B$.

   (b) Show that $A \subset f^{-1}(f(A))$. Show that equality holds if f is injective. Give an example to show that equality need not hold in general.

   (c) Show that $f(f^{-1}(B)) \subset B$. Show that equality holds if f is surjective. Give an example to show that equality need not hold in general.

5. What is one question you have about last week's lectures?