Homework 4

Due Friday, October 19, 2018

Do either 1 or 1$'$.

1. Let X be a topological space and $A \subset X$. The *closure* of A, denoted $\bar{A}$, is the intersection of all closed sets containing A.

   (a) Show that $\bar{A}$ is the smallest closed subset of X containing A, in the following sense: if $A \subset F \subset X$ and F is closed, then $\bar{A} \subset F$.

   (b) Show that if $A \subset B \subset X$ then $\bar{A} \subset \bar{B}$.

   (c) Show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

   (d) Show that $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. Give an example where the inclusion is strict.

1$'$. Let X be a topological space and $A \subset X$. The *interior* of A, denoted $\operatorname{int} A$, or sometimes A°, is the union of all open sets contained in A.

   (a) Show that $\operatorname{int} A$ is the biggest open subset of A, in the following sense: if $U \subset A$ and U is open, then $U \subset \operatorname{int} A$.

   (b) Show that if $A \subset B \subset X$ then $\operatorname{int} A \subset \operatorname{int} B$.

   (c) Show that $\operatorname{int} A \cap \operatorname{int} B = \operatorname{int}(A \cap B)$.

   (d) Show that $\operatorname{int} A \cup \operatorname{int} B \subset \operatorname{int}(A \cup B)$. Give an example where the inclusion is strict.

2. Let X be a topological space and $A \subset X$.

   (a) Show that $X \setminus \bar{A} = \operatorname{int}(X \setminus A)$, and $X \setminus \operatorname{int} A = \overline{X \setminus A}$.

   (b) The *boundary* of A, denoted ∂A, is defined to be $\bar{A} \setminus \operatorname{int} A$. Show that $\partial A = \partial(X \setminus A)$.

3. Find the closure, interior, and boundary of each subset of $\mathbb{R}^2$ (in the usual topology):

   (a) $A_1 = \{(x, y) : 0 < x \leq 1, 0 \leq y < 1\}$

   (b) $A_2 = \{(x, y) : 0 < x \leq 1, y = 0\}$

   (c) $A_3 = \{(x, y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$

4. Let X be a set, and let T be the set of subsets $U \subset X$ that such that $X \setminus U$ is finite, together with the empty set.

   (a) Show that T is a topology. (It is called the "finite complement topology.")

   (b) Find the closure, interior, and boundary of $\mathbb{Z}$ as a subset of $\mathbb{R}$ in the finite complement topology.

5. Let T be the set of subsets $U \subset \mathbb{R}$ such that U contains 0, together with the empty set.

   (a) Show that T is a topology.

   (b) Find the closure, interior, and boundary of the one-point subsets $\{1\}$ and $\{0\}$.

6. What is one question you have about last week's lectures?