Homework 5

Due Friday, October 26, 2018

1. The *lower semi-continuous topology* on $\mathbb{R}$ has the following open sets: $\varnothing$, all of $\mathbb{R}$, and sets of the form (a, ∞) for some $a \in \mathbb{R}$.

   (a) Show that the constant sequence $x_1 = x_2 = \cdots = 1$ converges to 1, and also to 0, but not to 2. That is, in the limit we can jump down, but not up.

   (This topology is not as silly as it might seem: the map that sends a matrix to its rank is lower semi-continuous, i.e. in the limit it can jump down but not up.)

   (b) Let X be $\mathbb{R}$ with the usual topology, let Y be $\mathbb{R}$ with the lower semi-continuous topology, and consider the functions $f, g \colon X \to Y$ defined by

   $$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x \geq 1 \\ 1 & \text{if } 0 < x < 1. \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 1 \\ 1 & \text{if } 0 \leq x \leq 1. \end{cases}$$

   Graph both functions. Which one is continuous?

2. Let X and Y be topological spaces. By definition, a map $f \colon X \to Y$ is *continuous* if for every open set $V \subset Y$, the preimage $f^{-1}(V)$ is open in X. What should it mean for f to be *continuous at a point* $p \in X$?

3. What is one question you have about last week's lectures?