Homework 6

Due Friday, November 2, 2018

Do either 3 or 3′. This week, $\mathbb{R}$ always has the usual topology.

1. Recall from lecture that if X is a topological space and $Y \subset X$, the *subspace topology* on Y is defined by saying that a subset $V \subset Y$ is open if and only if there is an open $U \subset X$ such that $U \cap Y = V$. Moreover, we proved that a subset $G \subset Y$ is closed in the subspace topology if and only if there is a closed $F \subset X$ such that $F \cap Y = G$.

   For the parts below, let $A \subset Y \subset X$.

   (a) Show that if A is closed in Y and Y is closed in X, then A is closed in X.

   (b) Give two examples to show that if A is closed in Y and Y is *not* closed in X, then A may or may not be closed in X.

   (c) Show that if A is open in Y and Y is open in X, then A is open in X.

   (d) Give two examples to show that if A is open in Y and Y is *not* open in X, then A may or may not be open in X.

   On the next midterm there will almost certainly be something about the closure and interior of A in Y versus in X.

2. (a) Let X be a topological space, and suppose that we can write $X = F_1 \cup \ldots \cup F_n$, where each F_i is closed. Let Y be another topological space, let $f\colon X \to Y$, and let $f_i\colon F_i \to Y$ be the restriction of f to F_i: that is, for $x \in F_i$ we set $f_i(x) = f(x)$.

   Show that f is continuous if and only if f_i is continuous for all i. Hint: Use the fact that a map is continuous if and only if the preimage of every closed set is closed.

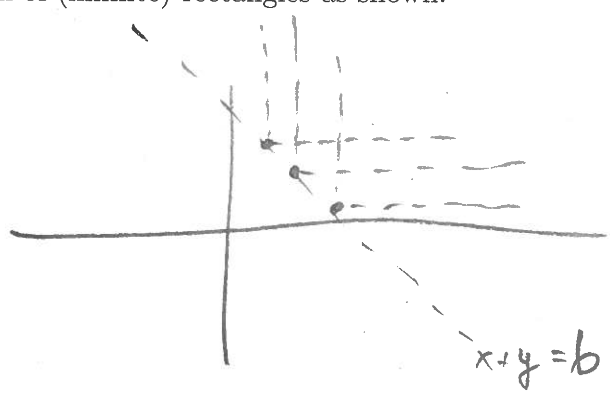
(b) This is usually applied to show that a piecewise function is continuous. Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1/3, \\ 3x - 1 & \text{if } 1/3 \leq x \leq 2/3, \\ 1 & \text{if } x \geq 2/3. \end{cases}$$

If we wanted to apply part (a) to show that f is continuous, what should sets should we take for the F_i?

3. Show that the addition map $a\colon \mathbb{R}^2 \to \mathbb{R}$, defined by $a(x,y) = x + y$, is continuous. Fill in the details of the following outline.

(a) For all $b \in \mathbb{R}$, the inverse image of the interval $(b, \infty) \subset \mathbb{R}$ is a union of (infinite) rectangles as shown:

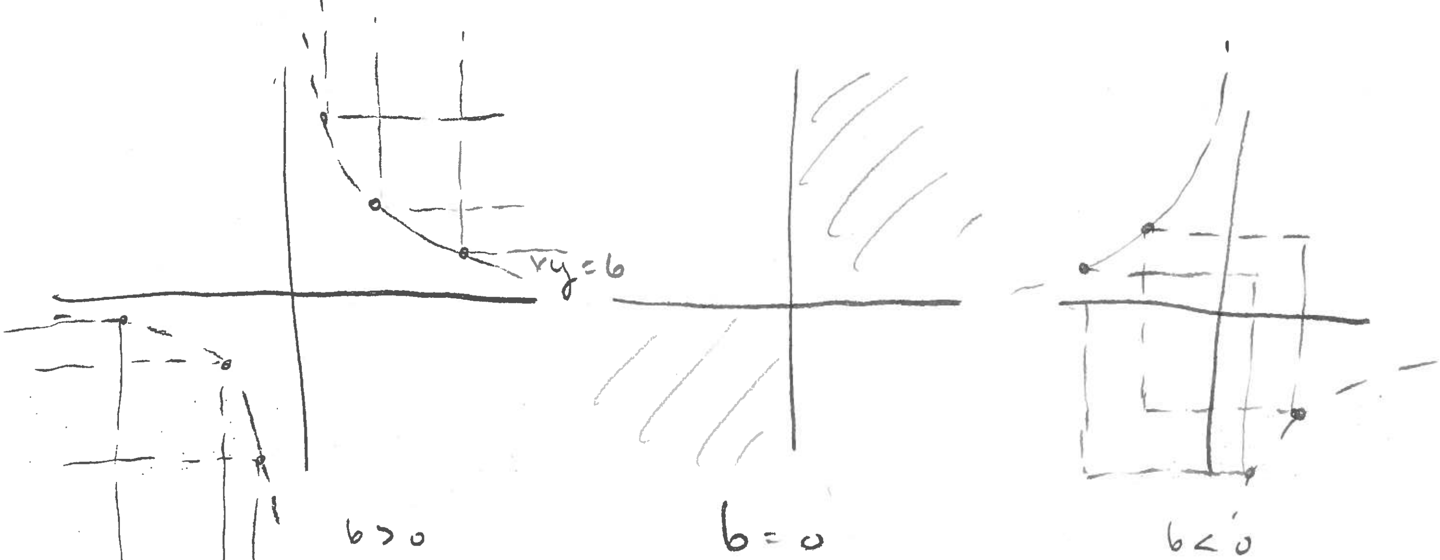


Hence $a^{-1}((b,\infty))$ is open, by definition of the product topology on $\mathbb{R}^2$.

(b) Similarly, $a^{-1}((-\infty, c))$ is open for all $c \in \mathbb{R}$. (Just copy this sentence and leave it at that.)

(c) If X is any topological space and $f\colon X \to \mathbb{R}$ a map with the property that $f^{-1}((b,\infty))$ and $f^{-1}((-\infty,c))$ are open for all $b, c \in \mathbb{R}$, then f is continuous.

3′. Same but with the multiplication map $m(x, y) = xy$. You will probably have to treat $b < 0$, $b = 0$, and $b > 0$ separately.



4. In lecture we considered the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

We saw that it was continuous as a function of x for each fixed y, and as a function of y for each fixed x, but not continuous as a function of x and y. Find an open set $U \subset \mathbb{R}$ such that $f^{-1}(U)$ is not open in $\mathbb{R}^2$.

Hint: To analyze f away from the origin, it may help to switch to polar coordinates, and to look up the formula for $\sin(2\theta)$.

5. What is one question you have about last week's lectures?