Homework 8

Due Friday, November 16, 2018

1. Let X be a topological space. Define a relation $\sim$ on X by declaring that $p \sim q$ iff there is a path from p to q, that is, a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.
   (a) Show that this is an equivalence relation: reflexive, symmetric, and transitive. Hint: The interesting one is transitive.
   (b) The equivalence classes of this equivalence relation are called are called *path components*. Describe (without proof) the path components of the following spaces:
      i. $\{(x, y) \in \mathbb{R}^2 : xy > 1\}$.
      ii. $\mathbb{Q}$.
      iii. The topologist's sine curve.
2. (a) Let X be a compact space, and let $F_1 \supset F_2 \supset F_3 \supset \cdots$ be a descending chain of non-empty closed subsets. Show that the intersection $F_1 \cap F_2 \cap F_3 \cap \cdots$ is not empty.
   Hint: Otherwise $X \setminus F_1$, $X \setminus F_2$, $\ldots$ is an open cover of X.
   (b) Give an example of a non-compact space X and a descending chain of closed subsets $F_1 \supset F_2 \supset F_3 \supset \cdots$ whose intersection is empty.
3. What is one question you have about last week's lectures?