

Final Exam

Tuesday, December 10, 2019

Each part is worth 10 points, for a total of 70 points.

1.
 - (a) Define what it means for a topological space X to be *disconnected*, and give two more equivalent conditions. (No proofs.)
 - (b) Show that $\mathbb{R}$ with the finite complement topology is connected.
 - (c) Let X be a topological space, let $A \subset X$, and let $Y \subset X$ be a connected subspace that intersects both A and $X \setminus A$. Show that Y intersects the boundary ∂A .
2. Do (a) and either (b) or (b').
 - (a) Define an *open cover* of a topological space X , and a *subcover*. Define what it means for a topological space X to be *compact*.
 - (b) Prove the following statement from the last homework: if $x \in X$, if Y is compact, and if $U \subset X \times Y$ is an open set containing $\{x\} \times Y$, then there is an open set $V \subset X$ with $x \in V$ and $V \times Y \subset U$.
 - (b') Prove the following statement that we proved in lecture: if X is Hausdorff, if $K \subset X$ is compact, and if $q \in X \setminus K$, then there are disjoint open sets $U, V \subset X$ with $K \subset U$ and $q \in V$.
3. Let X be a topological space, and let Φ be a set of continuous functions $X \rightarrow \mathbb{R}$ such that (i) for every $x \in X$ and every $f \in \Phi$ we have $f(x) \geq 0$, and (ii) for every $x \in X$ there is some $f \in \Phi$ with $f(x) > 0$.
 - (a) Show that if X is compact then there are $f_1, \dots, f_n \in \Phi$ such that $f_1(x) + \dots + f_n(x) > 0$ for all $x \in X$.
 - (b) Give a counterexample when X is not compact.