Homework 2

Due Monday, October 14, 2019

1. Let X, Y, and Z be metric spaces, with metrics d_X, d_Y, and d_Z. Let $f\colon X \to Y$ be continuous at a point $p \in X$, and let $g\colon Y \to Z$ be continuous at $f(p)$. Show that $g \circ f$ is continuous at p.

2. Let X be any set, and let d_X be the *discrete metric*

$$d_X(p,q) = \begin{cases} 0 & \text{if } p = q, \text{ or} \\ 1 & \text{if } p \neq q. \end{cases}$$

   (a) Show that this is a metric.

   (b) Let (Y, d_Y) be another metric space (not necessarily discrete). Show that every map $f\colon X \to Y$ is continuous.

3. Let $X = \mathbb{R}^2$ with the Euclidean metric. Sketch the subset

$$A = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \text{ or } y = 0\}.$$

   Show that A is neither open nor closed.

4. Let $X = \mathbb{Q}$ with the usual metric inherited from $\mathbb{R}$. Show that the set

$$A = \{x \in \mathbb{Q} : x^2 < 1\}$$

   is open but not closed, but that the set

$$B = \{x \in \mathbb{Q} : x^2 < 2\}$$

   is both open and closed. (You can use the fact that $\sqrt{2}$ is irrational without proving it.)

5. Optional: Let
$$W = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots, 0\}$$
with the usual metric inherited from $\mathbb{R}$. Let X be another metric space. Given a sequence $p_1, p_2, \dots \in X$ and a point $\ell \in X$, show that the function $f\colon W \to X$ defined by
$$\begin{cases} f(\tfrac{1}{n}) = p_n, \\ f(0) = \ell \end{cases}$$
is continuous if and only if $p_n \to \ell$.

6. What is one question you have about last week's lectures?