Homework 5

Due Monday, November 4, 2019

1. Let X be a topological space. Let $Y \subset X$, and give Y the subspace topology. Let $A \subset Y$.

   (a) Show that if A is closed in Y and Y is closed in X, then A is closed in X.

   (b) Give two examples to show that if A is closed in Y and Y is *not* closed in X, then A may or may not be closed in X.

   (c) Show that if A is open in Y and Y is open in X, then A is open in X.

   (d) Give two examples to show that if A is open in Y and Y is *not* open in X, then A may or may not be open in X.

   (e) Let $A \subset Y$, let $\mathrm{cl}_X(A)$ denote the closure of A in X, and let $\mathrm{cl}_Y(A)$ denote the closure of A in Y. Show that

   $$\mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap Y.$$

   (f) Let $A \subset Y$, let $\mathrm{int}_X(A)$ denote the interior of A as a subset of X, and let $\mathrm{int}_Y(A)$ denote the interior of A as a subset of Y. Show that

   $$\mathrm{int}_X(A) \subset \mathrm{int}_Y(A).$$

   (g) Give an example where the inclusion in part (f) is strict.

2. (a) Let X be a topological space, and suppose that we can write $X = F_1 \cup \ldots \cup F_n$, where each F_i is closed. Let Y be another topological space, let $f\colon X \to Y$, and let $f_i\colon F_i \to Y$ be the restriction of f to F_i: that is, for $x \in F_i$ we set $f_i(x) = f(x)$.

   Show that f is continuous if and only if f_i is continuous for all i. Hint: Use the fact that a map is continuous if and only if the preimage of every closed set is closed.

   (b) This is usually applied to show that a piecewise function is continuous. Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 1/3, \\ 3x - 1 & \text{if } 1/3 \leq x \leq 2/3, \\ 1 & \text{if } x \geq 2/3. \end{cases}$$

   If we wanted to apply part (a) to show that f is continuous, what should sets should we take for the F_i?

3. Because there's a midterm on Friday, let's take it easy and postpone questions on the product topology till next week. But don't forget last week's optional problem (read Moore's paper "The emergence of open sets…") and the two questions about it:

   (a) What is one thing you read that confused you?

   (b) What is one thing you read that surprised you?

4. What is one question you have about last week's lectures?