Homework 9

Due *Friday*, December 6, 2019

1. Which of the following topologies on $\mathbb{R}$ are compact? Give a proof either way.

   (a) The finite complement topology.

   (b) The topology where U is open iff either $U = \varnothing$ or $0 \in U$.

   (c) The lower semi-continuous topology.

2. (a) Let X be a compact space, and let $F_1 \supset F_2 \supset F_3 \supset \cdots$ be a descending chain of non-empty closed subsets. Show that the intersection $F_1 \cap F_2 \cap F_3 \cap \cdots$ is not empty.

   Hint: Otherwise $X \setminus F_1$, $X \setminus F_2$, $\ldots$ is an open cover of X.

   (b) Give an example of a non-compact space X and a descending chain of closed subsets $F_1 \supset F_2 \supset F_3 \supset \cdots$ whose intersection is empty.

3. (a) Let X and Y be topological spaces, let $A \subset X$, and let U be a neighborhood of $A \times Y$ in $X \times Y$. Show that if Y is compact then there is a neighborhood V of A in X such that $V \times Y \subset U$.

   (Start by drawing a picture!)

   (b) Give a counterexample when Y is not compact.

4. A continuous map $f\colon X \to Y$ is called *proper* if the preimage of any compact set $K \subset Y$ is compact.

   (a) Show that the map $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is proper.

   (b) Show that the map $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 - y^2$ is not proper.

(c) If f is proper then the preimage of every point is compact, because points are compact. But give an example of a continuous map $f\colon X \to Y$ for which the preimage of every point is compact, but nonetheless f is not proper.

(d) Show that if X is compact and Y is Hausdorff then any continuous map $f\colon X \to Y$ is proper.

(e) Let X and Y be topological spaces. Show that the projection $p\colon X \times Y \to X$ is proper if and only if Y is compact.

5. Optional. We define *one-point compactification* of a topological space X to be

$$\hat{X} = X \cup \{\infty\},$$

with the following topology: if U is open in X, then U is open in $\hat{X}$; and if $K \subset X$ is compact and closed, then $(X \setminus K) \cup \{\infty\}$ is open in $\hat{X}$. Equivalently, $F \subset \hat{X}$ is closed if $F \cap X$ is closed, and either (1) $\infty \in F$ or (2) F is compact.

(a) Show that the one-point compactification of $[0, 1)$ is homeomorphic to $[0, 1]$, and that the one-point compactification of $\mathbb{R}$ or $(0, 1)$ is homeomorphic to the circle. Describe the one-point compactification of $\mathbb{R}^2$.

(b) Show that $\hat{X}$ is compact.

(c) A space X is called *locally compact* if for every point $p \in X$ there is a compact set $K \subset X$ with $p \in \operatorname{int}(K)$. This is a slightly unusual use of the word "locally," but it the appropriate one for compactness. For example, $\mathbb{R}$ is locally compact, but $\mathbb{Q}$ is not.

Show that $\hat{X}$ is Hausdorff if and only if X is locally compact and Hausdorff.

(d) A map $f\colon X \to Y$ induces a map $\hat{f}\colon \hat{X} \to \hat{Y}$. Show that $\hat{f}$ is continuous if and only if f is proper.