Solutions to Final Exam

1. (a) Define what it means for a topological space $X$ to be connected.

**Solution:** $X$ is disconnected if we can write $X = U \cup V$, where $U$ and $V$ are open, disjoint, and non-empty. $X$ is connected if it is not disconnected.

Many equivalent formulations are also acceptable.

(b) Show that $X$ is disconnected if and only if there is a continuous surjection $f : X \to Z$, where $Z = \{1, 2\}$ with the discrete topology. (Recall that the discrete topology means every subset is open.)

**Solution:** First suppose that there is a continuous surjection $f : X \to Z$. Let $U = f^{-1}(\{1\})$ and $V = f^{-1}(\{2\})$. We have

$$X = f^{-1}(Z) = f^{-1}(\{1\} \cup \{2\}) = f^{-1}(\{1\}) \cup f^{-1}(\{2\}) = U \cup V.$$  

Because $\{1\}$ and $\{2\}$ are open in $Z$ and $f$ is continuous, $U$ and $V$ are open in $X$. Because $\{1\}$ and $\{2\}$ are disjoint and $f^{-1}$ preserves intersections, $U$ and $V$ are disjoint. Because $f$ is surjective, $U$ and $V$ are non-empty. Thus $X$ is disconnected.

Conversely, suppose that $X$ is disconnected, and write $X = U \cup V$, where $U$ and $V$ are open, disjoint, and non-empty. Define $f : X \to Z$ by

$$f(x) = \begin{cases} 
1 & \text{if } x \in U, \\
2 & \text{if } x \in V.
\end{cases}$$

Because $X = U \cup V$, $f$ is defined everywhere, and because $U$ and $V$ are disjoint, it is well-defined. Because $U$ and $V$ are non-empty,
it is surjective. To see that \( f \) is continuous, observe that

\[
\begin{align*}
  f^{-1}(\emptyset) &= \emptyset \\
  f^{-1}(\{1\}) &= U \\
  f^{-1}(\{2\}) &= V \\
  f^{-1}(\{1,2\}) &= X,
\end{align*}
\]

all of which are open.

(c) Use part (b) to show that if \( Y \) is a topological space and \( A, B \subset Y \) are non-empty, connected subspaces with a common point \( y \in A \cap B \), then \( A \cup B \) is connected.

(This was the key step in Homework 9 #1(a).)

**Solution:** Let \( Z = \{1,2\} \) with the discrete topology, and let \( f : A \cup B \to Z \) be continuous. We want to show that \( f \) is not surjective.

Suppose that \( f(y) = 1 \). Because \( A \) is connected, \( f|_A : A \to Z \) is not surjective; on the other hand, \( y \in A \), so \( f(y) = 1 \in f(A) \), so \( f(A) = \{1\} \). Similarly \( f(B) = \{1\} \). Thus \( f(A \cup B) = f(A) \cup f(B) = \{1\} \), so \( f \) is not surjective, as desired.

Similarly, if \( f(y) = 2 \) we find that \( f(A \cup B) = \{2\} \), so \( f \) is not surjective.

2. Let \( X \) and \( Y \) be topological spaces, and let \( p : X \times Y \to X \) be the projection \( p(x, y) = x \).

We have seen in lecture that \( p \) is continuous, and that \( p \) is an open map: that is, if \( W \subset X \times Y \) is open then \( p(W) \subset X \) is open.

(a) Give an example to show that \( p \) need not be a closed map: that is, if \( F \subset X \times Y \) is closed then \( p(F) \subset X \) need not be closed.

Hint: We’ve seen this in lecture; think about a hyperbola.

**Solution:** Let \( X = Y = \mathbb{R} \), and let

\[
F = \{(x, y) \in \mathbb{R}^2 : xy = 1\}.
\]

Then \( F \) is closed, but \( p(F) = (-\infty, 0) \cup (0, \infty) \) is not closed.
(b) Define what it means for \( Y \) to be compact.

**Solution:** Every open cover of \( Y \) admits a finite subcover: that is, if \( C \) is a collection of open sets in \( Y \) with \( \bigcup C = Y \), then there are finitely many \( U_1, \ldots, U_n \in C \) with \( Y = U_1 \cup \cdots \cup U_n \).

(c) Prove the following lemma that we proved in lecture: If \( Y \) is compact, \( W \subset X \times Y \) is open, and \( \{x\} \times Y \subset W \) for some point \( x \in X \), then there is an open set \( U \subset X \) with \( x \in U \) and \( U \times Y \subset W \).

Hint: Maybe start by drawing a picture to help you keep the notation straight.

**Solution:** For all \( y \in Y \) we have \((x, y) \in W\). Because \( W \) is open in the product topology, for each \( y \) we can choose open sets \( U_y \subset X \) and \( V_y \subset Y \) with \((x, y) \in U_y \times V_y \subset W\). The \( V_y \)'s form an open cover of \( Y \). Because \( Y \) is compact, we can extract a finite subcover \( V_{y_1}, \ldots, V_{y_n} \). Let \( U = U_{y_1} \cap \cdots \cap U_{y_n} \). Then

\[
U \times Y = \bigcup_{i=1}^{n} U_i \times V_i \subset \bigcup_{i=1}^{n} U_i \times V_i \subset W.
\]

(d) Show that if \( Y \) is compact then \( p \) is a closed map: that is, if \( F \subset X \times Y \) is closed then \( p(F) \subset X \) is closed.

Hint: Show that \( X \setminus p(F) \) is open by showing that if \( x \in X \setminus p(F) \) then there is an open \( U \subset X \) such that \( x \in U \subset X \setminus p(F) \). Apply part (c) with \( W = (X \times Y) \setminus F \).

**Solution:** Let us show that \( X \setminus p(F) \) is open, following the hint.

If \( x \in X \setminus p(F) \) then \( p^{-1}(\{x\}) \subset (X \times Y) \setminus F =: W \). We have \( p^{-1}(\{x\}) = \{x\} \times Y \), so by part (c) there is an open set \( U \subset X \) with \( x \in U \) and \( p^{-1}(U) = U \times Y \subset W \). Thus \( U \subset X \setminus p(F) \).

3. Let \( X \) and \( Y \) be topological spaces and let \( f : X \to Y \) be any map. We define the **graph** of \( f \),

\[
\Gamma = \{(x, y) \in X \times Y : y = f(x)\}.
\]

(a) Define what it means for \( Y \) to be Hausdorff. State (but do not prove) the characterization of Hausdorff in terms of the diagonal \( \Delta \subset Y \times Y \).
Solution: For every pair of points $p, q \in Y$ there are open sets $U, V \subset Y$ with $p \in U$, $q \in V$, and $U \cap V = \emptyset$. This is equivalent to saying that the diagonal

$$\Delta = \{(y_1, y_2) \in Y \times Y : y_1 = y_2\}$$

is closed.

(b) Show that if $Y$ is Hausdorff and $f$ is continuous then $\Gamma$ is closed. Hint: Consider the continuous map $\varphi: X \times Y \to Y \times Y$ given by $(x, y) \mapsto (f(x), y)$, and the diagonal $\Delta \subset Y \times Y$.

Solution: The product topology is cooked up so that $p$ and $q$ are continuous, and so that $\varphi: X \times Y \to Y \times Y$ is continuous if and only if the two components $X \times Y \to Y$ are continuous. The first component of $\varphi$ is $(x, y) \mapsto f(x)$, that is, $f \circ p$, which is continuous because $f$ and $p$ are continuous. The second component if $(x, y) \mapsto y$, which is continuous. Thus $\varphi$ is continuous.

We have $\Gamma = \varphi^{-1}(\Delta)$: indeed, $(x, y) \in \Gamma$ if and only if $f(x) = y$, which is true if and only if $(f(x), y) \in \Delta$. Because $Y$ is Hausdorff, $\Delta$ is closed. Because $\varphi$ is continuous, $\varphi^{-1}(\Delta)$ is closed.

(c) Consider the projections

$$X \times Y \xrightarrow{p} Y \xrightarrow{q} Y$$

given by $p(x, y) = x$ and $q(x, y) = y$.

Show that for any subset $B \subset Y$ we have

$$f^{-1}(B) = p(\Gamma \cap q^{-1}(B)).$$

Solution: Let $x \in X$.

If $x \in f^{-1}(B)$ then $f(x) \in B$, so the point $(x, f(x))$ is in $\Gamma$ and in $q^{-1}(B) = X \times B$, so $p(x, f(x)) = x$ is in $p(\Gamma \cap q^{-1}(B))$.

Conversely, if $x = p(x, y)$ for some $(x, y) \in \Gamma \cap q^{-1}(B)$, then because $(x, y) \in \Gamma$ we have $y = f(x)$, and because $(x, y) \in q^{-1}(B)$ we have $y = q(x, y) \in B$. Thus $f(x) \in B$, so $x \in f^{-1}(B)$. 

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(d) Show that if $Y$ is compact and $\Gamma$ is closed, then $f$ is continuous.

Hint: Show that the preimage of a closed set $G \subset Y$ is closed. Apply 3(c) and 2(d).

**Solution:** Let $G \subset Y$ be closed. Because $q$ is continuous, $q^{-1}(G)$ is closed, so $q^{-1}(G) \cap \Gamma$ is closed. Because $Y$ is compact, $p$ is a closed map by 2(d), so $f^{-1}(G) = p(\Gamma \cap q^{-1}(G))$ is closed.

(e) Give an example to show that if $Y$ is Hausdorff but not compact and $\Gamma$ is closed then $f$ need not be continuous.

Hint: $f$ should “blow up” somewhere.

**Solution:** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 
\frac{1}{x} & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}$$

We have seen that $f$ is not continuous. But the graph of $f$ is the union of a hyperbola and a point, hence is closed.

(f) Give an example to show that if $Y$ is compact but not Hausdorff and $f$ is continuous then $\Gamma$ need not be closed.

Hint: The identity map is easiest.

**Solution:** Let $Y$ be a two-point indiscrete space, which is compact but not Hausdorff. Let $X = Y$, and let $f$ be the identity map $f(x) = x$. Then the graph of $f$ is the diagonal $\Delta \subset Y \times Y$, which is not closed because $Y$ is not Hausdorff.