Solutions to Final Exam

1. (a) Define what it means for a topological space $X$ to be disconnected, and give two more equivalent conditions. (No proofs.)

**Solution:** We can write $X = U \cup V$, where $U$ and $V$ are open, disjoint, and non-empty.

Equivalently, we can write $X = F \cup G$, where $F$ and $G$ are closed, disjoint, and non-empty.

Equivalently, there is a proper subset $A \subset X$ that is non-empty, open, and closed.

Equivalently, there is a continuous surjection $X \rightarrow \{0, 1\}$ with the discrete topology on the latter.

(b) Show that $\mathbb{R}$ with the finite complement topology is connected.

**Solution:** Recall that a subset $F \subset \mathbb{R}$ is closed in this topology if either $F$ is finite or $F = \mathbb{R}$. If we could write $\mathbb{R} = F \cup G$ with $F$ and $G$ closed, disjoint, and non-empty, then we could not have $F = \mathbb{R}$ or $G = \mathbb{R}$, so both $F$ and $G$ would be finite, so $\mathbb{R}$ would be finite, which is absurd.

(c) Let $X$ be a topological space, let $A \subset X$, and let $Y \subset X$ be a connected subspace that intersects both $A$ and $X \setminus A$. Show that $Y$ intersects the boundary $\partial A$.

**Solution:** There are a few ways to do this. Here’s one.

We know that $\overline{A}$ and $X \setminus A$ are closed in $X$, so $F := Y \cap \overline{A}$ and $G := Y \cap X \setminus A$ are a closed in $Y$ by definition of the subspace topology. By hypothesis, $F$ and $G$ are non-empty. Recall that $\partial A = A \cap X \setminus \overline{A}$, so $Y \cap \partial A = F \cap G$. Thus if $Y \cap \partial A = \emptyset$ then $Y$ would be disconnected.
2. Do (a) and either (b) or (b').

(a) Define an open cover of a topological space \( X \), and a subcover. Define what it means for a topological space \( X \) to be compact.

**Solution:** A collection \( C \) of subsets of \( X \) is an open cover if each \( U \in C \) is open, and \( \bigcup C = X \). A subset \( S \subset C \) is a subcover if \( \bigcup S = X \). A space is compact if every open cover admits a finite subcover.

(If you talked about a subspace \( A \subset X \) and a cover of \( A \) by open subsets of \( X \), I'll accept it, assuming what you said is correct.)

(b) Prove the following statement from the last homework: if \( x \in X \), if \( Y \) is compact, and if \( U \subset X \times Y \) is an open set containing \( \{x\} \times Y \), then there is an open set \( V \subset X \) with \( x \in V \) and \( V \times Y \subset U \).

**Solution:** For each \( y \in Y \) we can choose open subsets \( V_y \subset X \) and \( W_y \subset Y \) with \( (x, y) \in V_y \times W_y \subset U \) by definition of the product topology. Then the \( W_y \)'s from an open cover of \( Y \), so we can extract a finite subcover \( W_{y_1}, \ldots, W_{y_n} \). Let \( V = V_{y_1} \cap \cdots \cap V_{y_n} \).

(You don't need to say more than this, but for a longer answer see my solutions to the last homework.)

(b') Prove the following statement that we proved in lecture: if \( X \) is Hausdorff, if \( K \subset X \) is compact, and if \( q \in X \setminus K \), then there are disjoint open sets \( U, V \subset X \) with \( K \subset U \) and \( q \in V \).

**Solution:** For each \( p \in K \) we can choose disjoint open sets \( U_p, V_p \subset X \) with \( p \in U_p \) and \( q \in V_p \). Then the \( U_p \)'s form an open cover of \( K \), so we can extract a finite subcover \( U_{p_1}, \ldots, U_{p_n} \). Let \( U = U_{p_1} \cup \cdots \cup U_{p_n} \) and \( V = V_{p_1} \cap \cdots \cap V_{p_n} \).

(Again, you could say more, but you don’t need to.)
3. Let $X$ be a topological space, and let $\Phi$ be a set of continuous functions $X \to \mathbb{R}$ such that (i) for every $x \in X$ and every $f \in \Phi$ we have $f(x) \geq 0$, and (ii) for every $x \in X$ there is some $f \in \Phi$ with $f(x) > 0$.

(a) Show that if $X$ is compact then there are $f_1, \ldots, f_n \in \Phi$ such that $f_1(x) + \cdots + f_n(x) > 0$ for all $x \in X$.

Solution: For each $f \in \Phi$, consider

$$U_f := f^{-1}((0, \infty)) = \{x \in X : f(x) > 0\},$$

which is open because $f$ is continuous. By hypothesis (ii) they cover $X$, so we can extract a finite subcover $U_{f_1}, \ldots, U_{f_n}$. Then for every $x \in X$ there is an $i \in \{1, \ldots, n\}$ such that $x \in U_{f_i}$, so $f_i(x)$ is positive, and every value $f_1(x), \ldots, f_n(x)$ is non-negative, so the sum $f_1(x) + \cdots + f_n(x)$ positive.

(b) Give a counterexample when $X$ is not compact.

Solution: Let $X = \mathbb{R}$ with the usual topology. For each $k \in \mathbb{Z}$, let

$$f_k(x) = \max(0, 1 - |x - k|),$$

which looks like this:

Let $\Phi = \{f_k : k \in \mathbb{Z}\}$.

For any $x \in \mathbb{R}$, let $k = \lfloor x \rfloor$, the greatest integer less than or equal to $x$; then we see that $x - k < 1$, so $f_k(x) > 0$.

But given $f_{k_1}, \ldots, f_{k_n} \in \Phi$, let $x = \min(k_1, \ldots, k_n) - 1$; then we see that $f_k(x) = 0$. 

3