Solutions to Homework 1

1. In lecture we introduced three standard metrics on $\mathbb{R}^2$: the Euclidean metric

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

the taxicab metric

$$d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

and the square metric

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

(a) For each of the three metrics, sketch the open ball of some radius $r > 0$ around the origin

$$B_r(0) = \{p \in \mathbb{R}^2 : d(p, 0) < r\}.$$

Solution:
(b) For one of the three metrics (your choice), prove or give a counterexample to the following statement: a sequence of points \( p_n = (x_n, y_n) \) converges to a limit \( p = (x, y) \) if and only if \( x_n \to x \) and \( y_n \to y \) separately, as sequences in \( \mathbb{R} \) with the usual metric.

**Solution:** The statement is true in all three metrics.

First suppose that \( x_n \to x \) and \( y_n \to y \) separately. Let \( \epsilon > 0 \) be given. Choose an integer \( N_1 \) such that \( |x_n - x| < \epsilon/2 \) for all \( n \geq N_1 \), and an integer \( N_2 \) such that \( |y_n - y| < \epsilon/2 \) for all \( n \geq N_2 \). Let \( N = \max\{N_1, N_2\} \), and suppose that \( n \geq N \). In the taxicab metric, we have

\[
d_1((x_n, y_n), (x, y)) = |x_n - x| + |y_n - y| < (\epsilon/2) + (\epsilon/2) = \epsilon.
\]

In the Euclidean metric, we have

\[
d_2((x_n, y_n), (x, y)) = \sqrt{|x_n - x|^2 + |y_n - y|^2} < \sqrt{(\epsilon/2)^2 + (\epsilon/2)^2} = \epsilon \cdot \sqrt{2}/2 < \epsilon.
\]

In the square metric, we have

\[
d_\infty((x_n, y_n), (x, y)) = \max\{|x_n - x|, |y_n - y|\} < \epsilon/2 < \epsilon.
\]

Thus \((x_n, y_n) \to (x, y)\) in all three metrics.

Conversely, suppose that \((x_n, y_n) \to (x, y)\) in any of the three metrics. In the taxicab metric we have

\[
|x_n - x| \leq |x_n - x| + |y_n - y| = d_1((x_n, y_n), (x, y)).
\]

In the Euclidean metric we have

\[
|x_n - x| = \sqrt{(x_n - x)^2} \leq \sqrt{|x_n - x|^2 + |y_n - y|^2} = d_2((x_n, y_n), (x, y)).
\]

In the square metric we have

\[
|x_n - x| \leq \max\{|x_n - x|, |y_n - y|\} = d_\infty((x_n, y_n), (x, y)).
\]

The right-hand sides go to zero as \( n \to \infty \), so \( |x_n - x| \to 0 \) as well, so \( x_n \to x \). Similarly \( y_n \to y \).
2. Consider the following silly metric on \( \mathbb{R}^2 \):

\[
    d((x_1, y_1), (x_2, y_2)) = \begin{cases} 
    |y_1 - y_2| & \text{if } x_1 = x_2 \\
    |y_1 - y_2| + 1 & \text{if } x_1 \neq x_2.
    \end{cases}
\]

(a) Prove that \( d \) is a metric, that is, it satisfies the three axioms.

**Solution:** Clearly \( d \) is symmetric, \( d(p, p) = 0 \), and \( d(p, q) > 0 \) if \( p \neq q \). It remains to check the triangle inequality:

\[
d(p, q) + d(q, r) \geq d(p, r).
\]

Write \( p = (x_1, y_1), q = (x_2, y_2), \) and \( r = (x_3, y_3). \) If \( x_1 = x_3 \) then we have

\[
d(p, q) + d(q, r) \geq |y_1 - y_2| + |y_2 - y_3| \geq |y_1 - y_3| = d(p, r).
\]

If \( x_1 \neq x_3 \) then either \( x_1 \neq x_2 \) or \( x_2 \neq x_3 \) or both. In any case we have

\[
d(p, q) + d(q, r) \geq |y_1 - y_2| + |y_2 - y_3| + 1 \geq |y_1 - y_3| + 1 = d(p, r).
\]

(b) Sketch the open balls of radius 1/2, 1, and 2 around the origin in this metric.
(c) Give an example of a sequence that converges in the Euclidean metric $d_2$ but not in our silly metric $d$.

Solution: Let $p_n = \left(\frac{1}{n}, 0\right)$. In the Euclidean metric we have $p_n \to (0, 0)$ by problem 1(b) above. But in the silly metric, we see that the sequence is not even Cauchy, because $d(p_m, p_n) = 1$ whenever $m \neq n$.

(d) Show that every sequence that converges in $d$ converges $d_2$.

Solution: Let $p_n = (x_n, y_n)$ be a sequence converging to a limit $\ell = (x, y)$ in the silly metric $d$. We want to show that $p_n \to \ell$ in the Euclidean metric $d_2$.

Let $\epsilon > 0$ be given. Because $p_n \to \ell$ in $d$, there is an $N$ such that $d(p_n, \ell) < \epsilon(1)$ for all $n \geq N$. Because $d(p_n, \ell) < 1$, we must have $x_n = x$, so

$$d_2(p_n, \ell) = \sqrt{(x_n - x)^2 + (y_n - y)^2} = |y_n - y| = d(p_n, \ell) < \epsilon.$$

3. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, let $\{p_n\}$ be a sequence in $X$ that converges to a point $\ell \in X$, and let $f: X \to Y$ be continuous at $\ell$. Show that the sequence $\{f(p_n)\}$ in $Y$ converges to $f(\ell)$.

Solution: Let $\epsilon > 0$ be given. Because $f$ is continuous at $\ell$, there is a $\delta > 0$ such that $d_Y(f(p_n), f(\ell)) < \epsilon$ whenever $d_X(p_n, \ell) < \delta$. Because $p_n$ converges to $\ell$, there is an integer $N$ such that $d_X(p_n, \ell) < \delta$ for all $n \geq N$. Thus $d_Y(f(p_n), f(\ell)) < \epsilon$ for all $n \geq N$.

4. Optional: In lecture we saw an example of a sequence in $C([0,1])$ that converges in the $L^1$ metric but not in the sup metric. Show that the reverse cannot happen: every sequence that converges in the sup metric converges in the $L^1$ metric.

Solution: Let $f_n \to g$ in the sup metric. Let

$$M_n = \sup_{x \in [0,1]} |f_n(x) - g(x)| = d_\infty(f_n, g).$$

Then $M_n \to 0$ as $n \to \infty$. Observe that $f_n(x) - g(x) \leq M_n$ for all $x \in [0, 1]$. Thus

$$d_1(f_n, g) = \int_0^1 |f_n(x) - g(x)| \, dx \leq \int_0^1 M_n \, dx = M_n,$$

which goes to zero as $n \to \infty$. Thus $f_n \to g$ in the $L^1$ metric.