Solutions to Homework 2

1. Let $X$, $Y$, and $Z$ be metric spaces, with metrics $d_X$, $d_Y$, and $d_Z$. Let $f: X \to Y$ be continuous at a point $p \in X$, and let $g: Y \to Z$ be continuous at $f(p)$. Show that $g \circ f$ is continuous at $p$.

**Solution:** Let $\epsilon > 0$ be given. Because $g$ is continuous at $f(p)$, there is an $\eta > 0$ such that $d_Y(f(p), r) < \eta$ implies $d_Z(g(f(p)), g(r)) < \epsilon$ for all $r \in Y$, and in particular for all $r$ of the form $g(q)$ for some $q \in X$. Because $f$ is continuous at $p$, there is a $\delta > 0$ such that $d_X(p, q) < \delta$ implies $d_Y(f(p), f(q)) < \eta$ for all $q \in X$. Thus $d_X(p, q) < \delta$ implies $d_Z(g(f(p)), g(f(q))) < \epsilon$ for all $q \in X$.

2. Let $(X, d)$ be a metric space, and fix a point $a \in X$. Show that the function $f: X \to \mathbb{R}$ given by $f(p) = d(p, a)$ is continuous.

(Hint: Use the triangle inequality.)

**Solution:** Let $p \in X$ and $\epsilon > 0$ be given. For all $q \in X$, the triangle inequality gives

$$d(p, a) \leq d(p, q) + d(q, a) \quad \text{and} \quad d(q, a) \leq d(q, p) + d(p, a),$$

which we can rewrite as

$$f(p) - f(q) \leq d(p, q) \quad \text{and} \quad f(q) - f(p) \leq d(q, p).$$

Thus

$$|f(p) - f(q)| \leq d(p, q).$$

Now take $\delta = \epsilon$. If $d(p, q) < \delta$ then

$$|f(p) - f(q)| \leq d(p, q) < \delta = \epsilon,$$

as required.
3. Let $X$ be any set, and let $d_X$ be the discrete metric

$$
d_X(p, q) = \begin{cases} 
0 & \text{if } p = q, \\
1 & \text{if } p \neq q.
\end{cases}
$$

(a) Show that this is a metric.

**Solution:** Clearly $d$ is symmetric, $d(p, p) = 0$ and $d(p, q) > 0$ if $p \neq q$. It remains to check the triangle inequality:

$$d(p, r) \leq d(p, q) + d(q, r)$$

for all $p, q, r \in X$. We could analyze a series of cases ($p = q = r$, or $p = q \neq r$, etc.) and see that the triangle inequality holds in each case. Or we could ask, how could the triangle inequality fail, knowing that all distances appearing in it are either 0 or 1? Only if the left-hand side is 1 and the right-hand side is 0. But if the right-hand side is 0 then $d_X(p, q) = d_X(q, r) = 0$, so $p = q$ and $q = r$, so $p = r$, so the left-hand side is 0, not 1.

(b) Let $(Y, d_Y)$ be another metric space (not necessarily discrete). Show that every map $f: X \to Y$ is continuous.

**Solution:** Let $p \in X$ and $\epsilon > 0$ be given, and let $\delta = \frac{1}{2}$. If $d_X(p, q) < \delta = \frac{1}{2}$ then $d_X(p, q) = 0$, so $p = q$, so $d_Y(f(p), f(q)) = 0 < \epsilon$.

4. Consider $\mathbb{R}^2$ with either the Euclidean, taxicab, or square metric (your choice). Prove or give a counterexample to the following statement: a sequence of points $p_n = (x_n, y_n)$ converges to a limit $p = (x, y)$ if and only if $x_n \to x$ and $y_n \to y$ separately, as sequences in $\mathbb{R}$ with the usual metric.

**Solution:** The statement is true.

First suppose that $x_n \to x$ and $y_n \to y$ separately. Let $\epsilon > 0$ be given. Choose an integer $N_1$ such that $|x_n - x| < \epsilon/2$ for all $n \geq N_1$, and an integer $N_2$ such that $|y_n - y| < \epsilon/2$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$, and suppose that $n \geq N$. In the taxicab metric, we have

$$d_1((x_n, y_n), (x, y)) = |x_n - x| + |y_n - y| < (\epsilon/2) + (\epsilon/2) = \epsilon.$$
In the Euclidean metric, we have
\[ d_2((x_n, y_n), (x, y)) = \sqrt{|x_n - x|^2 + |y_n - y|^2} < \sqrt{(\epsilon/2)^2 + (\epsilon/2)^2} = \epsilon \cdot \sqrt{2}/2 < \epsilon. \]

In the square metric, we have
\[ d_\infty((x_n, y_n), (x, y)) = \max\{|x_n - x|, |y_n - y|\} < \max\{\epsilon, \epsilon\} = \epsilon. \]
Thus \((x_n, y_n) \to (x, y)\) in all three metrics.

Conversely, suppose that \((x_n, y_n) \to (x, y)\) in any of the three metrics. In the taxicab metric we have
\[ |x_n - x| \leq |x_n - x| + |y_n - y| = d_1((x_n, y_n), (x, y)). \]
In the Euclidean metric we have
\[ |x_n - x| = \sqrt{(x_n - x)^2} \leq \sqrt{|x_n - x|^2 + |y_n - y|^2} = d_2((x_n, y_n), (x, y)). \]
In the square metric we have
\[ |x_n - x| \leq \max\{|x_n - x|, |y_n - y|\} = d_\infty((x_n, y_n), (x, y)). \]
In any case the right-hand side goes to zero as \(n \to \infty\), so \(|x_n - x| \to 0\) as well, so \(x_n \to x\). Similarly \(y_n \to y\).

5. Let \(p_n\) and \(q_n\) be two sequences in a metric space \((X, d)\), and let \(\ell \in X\). Suppose that \(p_n \to \ell\) as \(n \to \infty\), and \(d(p_n, q_n) \to 0\) as \(n \to \infty\). Show that \(q_n \to \ell\) as \(n \to \infty\).

**Solution:** Let \(\epsilon > 0\) be given. Choose an integer \(N_1\) such that \(d(p_n, \ell) < \epsilon/2\) for all \(n \geq N_1\), and an integer \(N_2\) such that \(d(p_n, q_n) \leq \epsilon/2\) for all \(n \geq N_2\), and let \(N = \max\{N_1, N_2\}\). If \(n \geq N\) then
\[ d(q_n, \ell) \leq d(q_n, p_n) + d(p_n, \ell) < \epsilon/2 + \epsilon/2 = \epsilon. \]

6. **Optional:** Let \(X = \mathbb{Q}\), let \(p\) be a prime number, and let \(d_p\) be the \(p\)-adic metric, defined as follows. Given \(x, y \in \mathbb{Q}\), if \(x = y\) then we define \(d_p(x, y) = 0\). If \(x \neq y\) then there is a unique integer \(n\) such that
\[ x - y = p^n \cdot \frac{a}{b}, \]
where \(a\) and \(b\) are integers not divisible by \(p\), and we define \(d_p(x, y) = p^{-n}\).
(a) Write down a few rational numbers and find the 2-adic distance between them.

Solution: I asked my computer to pick some random numbers, and it gave me $\frac{1}{2}$, 10, and $\frac{2}{3}$. First,

$$\frac{1}{2} - 10 = -\frac{19}{2} = 2^{-1} \cdot \frac{19}{1},$$

so $d_2\left(\frac{1}{2}, 10\right) = 2^1 = 2$. Next,

$$10 - \frac{2}{3} = \frac{28}{3} = 2^2 \cdot \frac{7}{3},$$

so $d_2\left(10, \frac{2}{3}\right) = 2^{-2} = \frac{1}{4}$. Last,

$$\frac{1}{2} - \frac{2}{3} = -\frac{1}{6} = 2^{-1} \cdot \frac{1}{3},$$

so $d_2\left(\frac{1}{2}, \frac{2}{3}\right) = 2^1 = 2$. As a sanity check, observe that these distances satisfy the triangle inequality.

(b) Show that the $p$-adic metric is a metric (for any prime $p$).

Solution: It is clear that $d(x, x) = 0$ and $d(x, y) > 0$ if $x \neq y$. To see that $d$ is symmetric, suppose that $x \neq y$, and write

$$x - y = p^n \cdot \frac{a}{b}$$

where $a$ and $b$ are not divisible by $p$. Then

$$y - x = p^n \cdot \frac{-a}{b},$$

so $d(x, y) = p^{-n} = d(y, x)$.

It remains to check the triangle inequality:

$$d(x, z) \leq d(x, y) + d(y, z)$$

Write

$$x - y = p^n \cdot \frac{a}{b} \quad \text{and} \quad y - z = p^m \cdot \frac{c}{d},$$

where $a$, $b$, $c$, and $d$ are not divisible by $p$, bearing in mind that $m$ and $n$ may be negative. If $m \geq n$ then

$$x - z = (x - y) + (y - z) = p^n \cdot \frac{ad + p^{m-n}bc}{bd}. $$
Because $b$ and $d$ are not divisible by $p$ and $p$ is prime, the denominator $bd$ is not divisible by $p$. But the numerator may be divisible by $p$, so write $ad + p^{m-n}bc = p^k \cdot e$ for some $k \geq 0$ and some $e$ not divisible by $p$. Then we have

$$x - z = p^{n+k} \cdot \frac{e}{bd}.$$  

Because $k \geq 0$ we have

$$d(x, z) = p^{-n-k} = p^{-n}p^{-k} \leq p^{-n} \leq p^{-n} + p^{-m} = d(x, y) + d(y, z),$$

as desired. If $m \leq n$ then the argument is similar.

(c) The last part still has issues, even with the correction. I might fix it later. Sorry.