Homework 3

Due Monday, October 21, 2019

1. Let $f\colon X \to Y$ be an arbitrary map of sets. The *image* of a subset $A \subset X$ is
$$f(A) = \{f(a) : a \in A\},$$
or equivalently
$$f(A) = \{y \in Y : y = f(a) \text{ for some } a \in A\}.$$
This is a subset of Y.

   (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f(A)$ for the following subsets $A \subset \mathbb{R}$: the intervals $[-1, 1]$, $[-1, 1)$, $(-1, 1)$, $[0, 1]$, $[0, 1)$, and $(0, 1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

   (b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x$. Choose a few subsets $A \subset \mathbb{R}^2$ and sketch both A and $f(A)$.

   (c) Now let $f\colon X \to Y$ be arbitrary, and let $A, B \subset X$. Show that if $A \subset B$ then $f(A) \subset f(B)$. Show that $f(A \cup B) = f(A) \cup f(B)$. Show that $f(A \cap B) \subset f(A) \cap f(B)$, but give an example where the two are not equal.

2. Let $f\colon X \to Y$ be an arbitrary map of sets. The *preimage* of a subset $B \subset Y$ is
$$f^{-1}(B) = \{x \in X : f(x) \in B\},$$
or equivalently
$$f^{-1}(B) = \{x \in X : f(x) = b \text{ for some } b \in B\}.$$
This is a subset of X.

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find $f^{-1}(B)$ for the following subsets $B \subset \mathbb{R}$: the intervals $[-1,1]$, $[-1,1)$, $(-1,1)$, $[0,1]$, $[0,1)$, and $(0,1)$, and the singletons $\{-1\}$, $\{0\}$, and $\{1\}$.

(b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = xy$. Sketch $f^{-1}(B)$ when B is $\{0\}$, $\{1\}$, $[0,1]$, $(0,1)$, and $[0,1)$.

(c) Now let $f\colon X \to Y$ be arbitrary, and let $A, B \subset Y$. Show that if $A \subset B$ then $f^{-1}(A) \subset f^{-1}(B)$. Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, and that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

3. Let $f\colon X \to Y$, let $A \subset X$, and let $B \subset Y$.

(a) Show that $A \subset f^{-1}(B)$ if and only if $f(A) \subset B$.

(b) Show that $A \subset f^{-1}(f(A))$. Show that equality holds if f is injective. Give an example to show that equality need not hold in general.

(c) Show that $f(f^{-1}(B)) \subset B$. Show that equality holds if f is surjective. Give an example to show that equality need not hold in general.

4. (a) The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases}$$

is discontinuous (with the usual topology on $\mathbb{R}$). Exhibit an open set $V \subset \mathbb{R}$ such that $f^{-1}(V)$ is not open.

(b) Optional: The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is discontinuous (with the usual topology on $\mathbb{R}^2$ and $\mathbb{R}$). Exhibit an open set $V \subset \mathbb{R}$ such that $f^{-1}(V)$ is not open.

5. Optional: A map $f\colon X \to Y$ is called *open* if for every open set $U \subset X$, the image $f(U)$ is open in Y.

   (a) Give an example of a continuous map that is not open.

   (b) Give an example of an open map that is not continuous.

   (c) Give an example of a map that is both continuous and open.

   (d) Give an example of a map that is neither continuous nor open.

6. What is one question you have about last week's lectures?