Homework 4

Due Monday, October 28, 2019

Do either 1 or 1′. But read them both.

1. Let X be a topological space and $A \subset X$. The *closure* of A, denoted $\bar{A}$, is the intersection of all closed sets containing A.

   (a) Show that $\bar{A}$ is the smallest closed subset of X containing A, in the following sense: if $A \subset F \subset X$ and F is closed, then $\bar{A} \subset F$.

   (b) Show that if $A \subset B \subset X$ then $\bar{A} \subset \bar{B}$.

   (c) Show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

   (d) Show that $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. Give an example where the inclusion is strict.

1′. Let X be a topological space and $A \subset X$. The *interior* of A, denoted $\operatorname{int} A$, or sometimes A°, is the union of all open sets contained in A.

   (a) Show that $\operatorname{int} A$ is the biggest open subset of A, in the following sense: if $U \subset A$ and U is open, then $U \subset \operatorname{int} A$.

   (b) Show that if $A \subset B \subset X$ then $\operatorname{int} A \subset \operatorname{int} B$.

   (c) Show that $\operatorname{int} A \cap \operatorname{int} B = \operatorname{int}(A \cap B)$.

   (d) Show that $\operatorname{int} A \cup \operatorname{int} B \subset \operatorname{int}(A \cup B)$. Give an example where the inclusion is strict.

2. Let X be a topological space and $A \subset X$.

   (a) Show that $X \setminus \bar{A} = \operatorname{int}(X \setminus A)$, and $X \setminus \operatorname{int} A = \overline{X \setminus A}$.

   (b) The *boundary* of A, denoted ∂A, is defined to be $\bar{A} \setminus \operatorname{int} A$. Show that $\partial A = \partial(X \setminus A)$.

3. Find the closure, interior, and boundary of each subset of $\mathbb{R}^2$ (in the usual topology):

    (a) $A_1 = \{(x,y) : 0 < x \leq 1, 0 \leq y < 1\}$

    (b) $A_2 = \{(x,y) : 0 < x \leq 1, y = 0\}$

    (c) $A_3 = \{(x,y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$

Do two of the next three problems. Optional: Do all three.

4. Let X be a set, and let T be the set of subsets $U \subset X$ that such that $X \setminus U$ is finite, together with the empty set.

    (a) Show that T is a topology. (It is called the "finite complement topology.")

    (b) Find the closure, interior, and boundary of $\mathbb{Z}$ as a subset of $\mathbb{R}$ in the finite complement topology.

5. Let T be the set of subsets $U \subset \mathbb{R}$ such that U contains 0, together with the empty set.

    (a) Show that T is a topology.

    (b) Find the closure, interior, and boundary of the one-point subsets $\{1\}$ and $\{0\}$.

6. Let T be the subsets of $\mathbb{R}$ of the form (a, ∞) for some $a \in \mathbb{R}$, together with the empty set and the whole set $\mathbb{R}$.

    (a) Show that T is a topology. (It is called the "lower semi-continuous topology" and we discussed it in lecture on Friday.)

    (b) Find the closure, interior, and boundary of the interval $(0,1)$ as a subset of $\mathbb{R}$ in this topology.

7. Optional, due in two weeks (11/4): Read "The emergence of open sets, closed sets, and limit points in analysis and topology" by Gregory H. Moore, which is linked on Canvas and on the course web page.

    (a) What is one thing you read that confused you?

    (b) What is one thing you read that surprised you?

8. What is one question you have about last week's lectures?