1. Let $X$ be a topological space and $A \subset X$. The closure of $A$, denoted $\bar{A}$, is the intersection of all closed sets containing $A$.

(a) Show that $\bar{A}$ is the smallest closed subset of $X$ containing $A$, in the following sense: if $A \subset F \subset X$ and $F$ is closed, then $\bar{A} \subset F$.

**Solution:** By definition, $\bar{A}$ is the intersection of all closed sets containing $A$: that is, $x \in \bar{A}$ if and only if $x \in F$ for all closed sets $F \subset X$ with $A \subset F$. Thus if $F \subset X$ is some closed set with $A \subset F$, then for all $x \in \bar{A}$ we have $x \in F$; thus $\bar{A} \subset F$.

It may be worth remarking that $\bar{A}$ is a closed set containing $A$: it’s an intersection of closed sets, hence is closed, and it’s an intersection of sets containing $A$, hence contains $A$. (If $x \in A$ then $x \in F$ for all closed sets $F \subset X$ with $A \subset F$, so $x$ is in the intersection of all such $F$, which is $\bar{A}$.)

(b) Show that if $A \subset B \subset X$ then $\bar{A} \subset \bar{B}$.

**Solution:** We have $A \subset B \subset \bar{B}$, so $\bar{B}$ is a closed set containing $A$, so by part (a) we get $\bar{A} \subset \bar{B}$.

(c) Show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

**Solution:** We have $A \subset A \cup B$, so $\bar{A} \subset \overline{A \cup B}$ by part (b), and similarly $\bar{B} \subset \overline{A \cup B}$, so $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

For the reverse inclusion, we have $A \subset \bar{A} \subset \overline{A \cup B}$, and $B \subset \bar{B} \subset \overline{A \cup B}$, so $A \cup B \subset \overline{A \cup B}$. But $\overline{A \cup B}$ is a union of two closed sets, hence is closed, so by part (a) we get $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. 

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(d) Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Give an example where the inclusion is strict.

**Solution:** We have $A \cap B \subset A \subset \overline{A}$, and $A \cap B \subset B \subset \overline{B}$, so $A \cap B \subset \overline{A} \cap \overline{B}$. The latter is an intersection of closed sets, hence is closed, so by part (a) we get $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

For the counterexample, let $X = \mathbb{R}$ with the usual topology, let $A = (0, 1)$, and let $B = (1, 2)$. Then $A \cap B = \emptyset$, which is closed, so $\overline{A \cap B} = \emptyset$. On the other hand, $\overline{A} = [0, 1]$, and $\overline{B} = [1, 2]$, so $\overline{A} \cap \overline{B} = \{1\}$.

1'. Let $X$ be a topological space and $A \subset X$. The *interior* of $A$, denoted $\text{int} A$, or sometimes $A^\circ$, is the union of all open sets contained in $A$.

(a) Show that $\text{int} A$ is the biggest open subset of $A$, in the following sense: if $U \subset A$ and $U$ is open, then $U \subset \text{int} A$.

**Solution:** By definition, $\text{int} A$ is the union of all open sets contained $A$: that is, $x \in \text{int} A$ if and only if $x \in U$ for some open set $U \subset A$. Thus if $U \subset X$ is some open set with $U \subset A$, then for all $x \in U$ we have $x \in \text{int} A$; thus $U \subset \text{int} A$.

It may be worth remarking that $\text{int} A$ is an open set contained in $A$: it’s a union of open sets, hence is open, and it’s a union of sets contained in $A$, hence is contained in $A$. (If $x \in \text{int} A$ then $x \in U$ for some open set $U \subset A$, so $x \in A$.)

(b) Show that if $A \subset B \subset X$ then $\text{int} A \subset \text{int} B$.

**Solution:** We have $\text{int} A \subset A \subset B$, so $\text{int} A$ is an open set contained in $B$, so by part (a) we get $\text{int} A \subset \text{int} B$.

(c) Show that $\text{int} A \cap \text{int} B = \text{int}(A \cap B)$.

**Solution:** We have $A \cap B \subset A$, so $\text{int}(A \cap B) \subset \text{int} A$ by part (b), and similarly $\text{int}(A \cap B) \subset \text{int} B$; thus $\text{int}(A \cap B) \subset \text{int} A \cap \text{int} B$.

For the reverse inclusion, we have $\text{int} A \cap \text{int} B \subset \text{int} A \subset A$, and $\text{int} A \cap \text{int} B \subset \text{int} B \subset B$, so $\text{int} A \cap \text{int} B \subset A \cap B$. But $\text{int} A \cap \text{int} B$ is an intersection of two open sets, hence is open, so by part (a) we get $\text{int} A \cap \text{int} B \subset \text{int}(A \cap B)$.
(d) Show that \( \text{int} A \cup \text{int} B \subset \text{int}(A \cup B) \). Give an example where the inclusion is strict.

**Solution:** We have \( \text{int} A \subset A \subset A \cup B \), and \( \text{int} B \subset B \subset A \cup B \). Now \( \text{int} A \cup \text{int} B \) is a union of open sets, hence is open, so by part (a) we get \( \text{int} A \cup \text{int} B \subset \text{int}(A \cup B) \).

For the counterexample, let \( X = \mathbb{R} \) with the usual topology, let \( A = (0, 1] \), and let \( B = [1, 2) \). Then \( \text{int} A = (0, 1) \), and \( \text{int} B = (1, 2) \), so \( \text{int} A \cup \text{int} B = (0, 1) \cup (1, 2) \). On the other hand, \( A \cup B = (0, 2) \) is open, so \( \text{int}(A \cup B) = (0, 2) \).

2. Let \( X \) be a topological space and \( A \subset X \).

(a) Show that \( X \setminus \bar{A} = \text{int}(X \setminus A) \), and \( X \setminus \text{int} A = \overline{X \setminus A} \).

**Solution:** It may be clearer to use the notation \( A^c := X \setminus A \) for the complement.

We first remark that for any subsets \( B, C \subset X \), we have \( B \subset C \) if and only if \( C^c \subset B^c \); and \( (B^c)^c = B \). You can prove these if you want.

Next, we have \( A \subset \bar{A} \), so \( (\bar{A})^c \subset A^c \); because \( \bar{A} \) is closed, \( (\bar{A})^c \) is open, so

\[
(\bar{A})^c \subset \text{int}(A^c)
\]

by \( 1'(a) \).

Replacing \( A \) with \( A^c \), we get \( (\bar{A^c})^c \subset \text{int}((A^c)^c) = \text{int} A \), so

\[
(\text{int} A)^c \subset \overline{A^c}.
\]

Thus we’ve proved one inclusion for each of the two desired equalities. For the reverse inclusions, we have \( \text{int} A \subset A \), so \( A^c \subset (\text{int} A)^c \); because \( \text{int} A \) is open, \( (\text{int} A)^c \) is closed, so

\[
\overline{A^c} \subset (\text{int} A)^c.
\]

Replacing \( A \) with \( A^c \), we get \( \bar{A} \subset (\text{int}(A^c))^c \), so

\[
\text{int}(A^c) \subset (\bar{A})^c
\]

by \( 1(a) \).

Thus both equalities are proved.
(b) The boundary of $A$, denoted $\partial A$, is defined to be $\bar{A} \setminus \text{int} \, A$. Show that $\partial A = \partial(\mathbb{X} \setminus A)$.

**Solution:** We have

$$\partial A = \bar{A} \setminus \text{int} \, A = \bar{A} \cap (\text{int} \, A)^c = \bar{A} \cap \mathbb{A}^c,$$

the last equality by part (a). The last expression is clearly symmetric between $A$ and $A^c$.

3. Find the closure, interior, and boundary of each subset of $\mathbb{R}^2$ (in the usual topology):

(a) $A_1 = \{(x,y) : 0 < x \leq 1, 0 \leq y < 1\}$

**Solution:** The interior is the open square given by $0 < x < 1$ and $0 < y < 1$. The closure is the closed square $0 \leq x \leq 1$ and $0 \leq y \leq 1$. The boundary is the four line segments shown:

![Diagram of a square with line segments](image)

(b) $A_2 = \{(x,y) : 0 < x \leq 1, y = 0\}$

**Solution:** The interior is empty. The closure is the line segment given by $0 \leq x \leq 1$ and $y = 0$. The boundary is the same line segment.

(c) $A_3 = \{(x,y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$

**Solution:** The interior is empty. The closure is all of $\mathbb{R}^2$. The boundary is all of $\mathbb{R}^2$. 

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4. Let $X$ be a set, and let $T$ be the set of subsets $U \subset X$ that such that $X \setminus U$ is finite, together with the empty set.

(a) Show that $T$ is a topology. (It is called the “finite complement topology.”)

**Solution:** We have $\emptyset \in T$ by definition. To see that $X \in T$, note that $X \setminus X = \emptyset$ is finite.

Suppose that $U_1, \ldots, U_n \in T$. If some $U_i = \emptyset$ then $U_1 \cap \cdots \cap U_n = \emptyset \in T$. Otherwise $X \setminus U_i$ is finite for all $i$, so $X \setminus (U_1 \cap \cdots \cap U_n) = (X \setminus U_1) \cup \cdots \cup (X \setminus U_n)$ is a finite union of finite sets, hence is finite, so $U_1 \cap \cdots \cap U_n \in T$.

Given an arbitrary subset $S \subset T$, we want to show that $\bigcup S \in T$. If $S = \emptyset$ or $S = \{\emptyset\}$ then $\bigcup S = \emptyset \in T$. Otherwise there is some non-empty $U \in S$, so $X \setminus U$ is finite. Then $U \subset \bigcup S$, so $X \setminus \bigcap S \subset X \cap U$ is a subset of a finite set, hence is finite, so $\bigcup S \in T$.

(Or it might be nicer to show that the closed sets do what they should.)

(b) Find the closure, interior, and boundary of $\mathbb{Z}$ as a subset of $\mathbb{R}$ in the finite complement topology.

**Solution:** A set is open if and only if its complement is finite, or it is empty. Thus a set is closed if and only if it is finite, or is the whole space $\mathbb{R}$.

Now $\mathbb{Z}$ is infinite, so the only closed set containing $\mathbb{Z}$ is $\mathbb{R}$, so $\bar{\mathbb{Z}} = \mathbb{R}$. Similarly, $\mathbb{R} \setminus \mathbb{Z}$ is infinite, so $\overline{\mathbb{R} \setminus \mathbb{Z}} = \mathbb{R}$, so $\text{int} \, \mathbb{Z} = \emptyset$ by 3(a). Finally, $\partial \mathbb{Z} = \bar{\mathbb{Z}} \setminus \text{int} \, \mathbb{Z} = \mathbb{R} \setminus \emptyset = \mathbb{R}$.
5. Let $T$ be the set of subsets $U \subset \mathbb{R}$ such that $U$ contains 0, together with the empty set.

(a) Show that $T$ is a topology.

**Solution:** We have $\emptyset \in T$ by definition, and $\mathbb{R} \in T$ because $0 \in \mathbb{R}$. 

Suppose that $U_1, \ldots, U_n \in T$. If some $U_i = \emptyset$ then $U_1 \cap \cdots \cap U_n = \emptyset \in T$. Otherwise $0 \in U_i$ for all $i$, so $0 \in U_1 \cap \cdots \cap U_n$, so $U_1 \cap \cdots \cap U_n \in T$. (Indeed, we can see that $T$ is closed under arbitrary intersections, not just finite intersections.)

Given an arbitrary subset $S \subset T$, we want to show that $\bigcup S \in T$. If $S = \emptyset$ or $S = \{\emptyset\}$ then $\bigcup S = \emptyset \in T$. Otherwise there is some non-empty $U \in S$, so $0 \in U$, so $0 \in \bigcup S$, so $\bigcup S \in T$.

(b) Find the closure, interior, and boundary of the one-point subsets \{1\} and \{0\}.

**Solution:** A set is open if and only if it either contains 0, or is empty. Thus a set is closed if and only if it either does not contain 0, or is the whole space $\mathbb{R}$.

Thus \{1\} is closed, and it contains no non-empty open set, so its interior is $\emptyset$, its closure is \{1\}, and its boundary is \{1\}, just as in the usual topology.

On the other hand, \{0\} is open, and it is not contained in any closed set apart from the whole space $\mathbb{R}$, so its interior is \{0\}, its closure is $\mathbb{R}$, and its boundary is $(-\infty, 0) \cup (0, \infty)$, quite unlike the usual topology.
6. Let $T$ be the subsets of $\mathbb{R}$ of the form $(a, \infty)$ for some $a \in \mathbb{R}$, together with the empty set and the whole set $\mathbb{R}$.

(a) Show that $T$ is a topology. (It is called the “lower semi-continuous topology” and we discussed it in lecture on Friday.)

Solution: We have $\emptyset \in T$ and $\mathbb{R} \in T$ by definition.

For finite intersections, by induction it is enough to prove it for two: if $U, V \in T$ then $U \cap V \in T$. Observe moreover that for any two subsets we either have $U \subset V$, in which case $U \cap V = U \in T$, or $V \subset U$, in which case $U \cap V = V \in T$.

Given an arbitrary subset $S \subset T$, we want to show that $\bigcup S \in T$. If $\mathbb{R} \in S$ then $\bigcup S = \mathbb{R} \in T$. If not, write

$$S = \{(a, \infty) : a \in A\}$$

or

$$S = \{(a, \infty) : a \in A\} \cup \{\emptyset\}$$

for some set $A$ of real numbers. If $A$ is non-empty and bounded below then $\bigcup S = (\inf A, \infty) \in T$. If $A$ is non-empty and unbounded below then $\bigcup S = \mathbb{R} \in T$. If $A$ is empty then $\bigcup S = \emptyset \in T$.

(b) Find the closure, interior, and boundary of the interval $(0, 1)$ as a subset of $\mathbb{R}$ in this topology.

Solution: The closed sets are of the form $(-\infty, a]$ for $a \in \mathbb{R}$, together with the whole set $\mathbb{R}$ and the empty set.

A closed set containing $(0, 1)$ is either all of $\mathbb{R}$, or of the form $(-\infty, a]$ for some $a \geq 1$. The closure of $(0, 1)$ is the intersection of all these, which is $(-\infty, 1]$. The only open set contained in $(0, 1)$ is $\emptyset$, so its interior is $\emptyset$. Thus its boundary is $(-\infty, 1]$. 