Solutions to Homework 4

1. Let $X$ be a topological space and $A \subset X$. The closure of $A$, denoted $\bar{A}$, is the intersection of all closed sets containing $A$.

   (a) Show that $\bar{A}$ is the smallest closed subset of $X$ containing $A$, in the following sense: if $A \subset F \subset X$ and $F$ is closed, then $\bar{A} \subset F$.

   **Solution:** By definition, $\bar{A}$ is the intersection of all closed sets containing $A$: that is, $x \in \bar{A}$ if and only if $x \in F$ for all closed sets $F \subset X$ with $A \subset F$. Thus if $F \subset X$ is some closed set with $A \subset F$, then for all $x \in \bar{A}$ we have $x \in F$; thus $\bar{A} \subset F$.

   It may be worth remarking that $\bar{A}$ is a closed set containing $A$: it's an intersection of closed sets, hence is closed, and it's an intersection of sets containing $A$, hence contains $A$. (If $x \in A$ then $x \in F$ for all closed sets $F \subset X$ with $A \subset F$, so $x$ is in the intersection of all such $F$, which is $\bar{A}$.)

   (b) Show that if $A \subset B \subset X$ then $\bar{A} \subset \bar{B}$.

   **Solution:** We have $A \subset B \subset \bar{B}$, so $\bar{B}$ is a closed set containing $A$, so by part (a) we get $\bar{A} \subset \bar{B}$.

   (c) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   **Solution:** We have $A \subset A \cup B$, so $\bar{A} \subset \overline{A \cup B}$ by part (b), and similarly $\bar{B} \subset \overline{A \cup B}$, so $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

   For the reverse inclusion, we have $A \subset \bar{A} \subset \overline{A \cup B}$, and $B \subset \bar{B} \subset \overline{A \cup B}$, so $A \cup B \subset \overline{A \cup B}$. But $\overline{A \cup B}$ is a union of two closed sets, hence is closed, so by part (a) we get $\overline{A \cup B} \subset \overline{A \cup B}$. 

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(d) Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Give an example where the inclusion is strict.

**Solution:** We have $A \cap B \subset A \subset \overline{A}$, and $A \cap B \subset B \subset \overline{B}$, so $A \cap B \subset \overline{A} \cap \overline{B}$. The latter is an intersection of closed sets, hence is closed, so by part (a) we get $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

For the counterexample, let $X = \mathbb{R}$ with the usual topology, let $A = (0, 1)$, and let $B = (1, 2)$. Then $A \cap B = \emptyset$, which is closed, so $\overline{A \cap B} = \emptyset$. On the other hand, $\overline{A} = [0, 1]$, and $\overline{B} = [1, 2]$, so $\overline{A \cap B} = \{1\}$.

1'. Let $X$ be a topological space and $A \subset X$. The **interior** of $A$, denoted int $A$, or sometimes $A^\circ$, is the union of all open sets contained in $A$.

(a) Show that int $A$ is the biggest open subset of $A$, in the following sense: if $U \subset A$ and $U$ is open, then $U \subset \text{int } A$.

**Solution:** By definition, int $A$ is the union of all open sets contained in $A$: that is, $x \in \text{int } A$ if and only if $x \in U$ for some open set $U \subset A$. Thus if $U \subset X$ is some open set with $U \subset A$, then for all $x \in U$ we have $x \in \text{int } A$; thus $U \subset \text{int } A$.

It may be worth remarking that int $A$ is an open set contained in $A$: it’s a union of open sets, hence is open, and it’s a union of sets contained in $A$, hence is contained in $A$. (If $x \in \text{int } A$ then $x \in U$ for some open set $U \subset A$, so $x \in A$.)

(b) Show that if $A \subset B \subset X$ then int $A \subset \text{int } B$.

**Solution:** We have int $A \subset A \subset B$, so int $A$ is an open set contained in $B$, so by part (a) we get int $A \subset \text{int } B$.

(c) Show that $\text{int } A \cap \text{int } B = \text{int}(A \cap B)$.

**Solution:** We have $A \cap B \subset A$, so int$(A \cap B) \subset \text{int } A$ by part (b), and similarly int$(A \cap B) \subset \text{int } B$; thus int$(A \cap B) \subset \text{int } A \cap \text{int } B$.

For the reverse inclusion, we have int $A \cap \text{int } B \subset \text{int } A \subset A$, and int $A \cap \text{int } B \subset \text{int } B \subset B$, so int $A \cap \text{int } B \subset A \cap B$. But int $A \cap \text{int } B$ is an intersection of two open sets, hence is open, so by part (a) we get int $A \cap \text{int } B \subset \text{int}(A \cap B)$. 

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(d) Show that \( \text{int } A \cup \text{int } B \subset \text{int}(A \cup B) \). Give an example where the inclusion is strict.

**Solution:** We have \( \text{int } A \subset A \subset A \cup B, \) and \( \text{int } B \subset B \subset A \cup B, \) so \( A \cup \text{int } B \subset A \cup B \). Now \( \text{int } A \cup \text{int } B \) is a union of open sets, hence is open, so by part (a) we get \( \text{int } A \cup \text{int } B \subset \text{int}(A \cup B) \).

For the counterexample, let \( X = \mathbb{R} \) with the usual topology, let \( A = (0, 1] \), and let \( B = [1, 2) \). Then \( \text{int } A = (0, 1) \), and \( \text{int } B = (1, 2) \), so \( \text{int } A \cup \text{int } B = (0, 1) \cup (1, 2) \). On the other hand, \( A \cup B = (0, 2) \) is open, so \( \text{int}(A \cup B) = (0, 2) \).

2. Let \( X \) be a topological space and \( A \subset X \).

(a) Show that \( X \setminus \bar{A} = \text{int}(X \setminus A) \), and \( X \setminus \text{int } A = \overline{X \setminus A} \).

**Solution:** It may be clearer to use the notation \( A^c := X \setminus A \) for the complement.

We first remark that for any subsets \( B, C \subset X \), we have \( B \subset C \) if and only if \( C^c \subset B^c \); and \( (B^c)^c = B \). You can prove these if you want.

Next, we have \( A \subset \bar{A} \), so \( (\bar{A})^c \subset A^c \); because \( \bar{A} \) is closed, \( (\bar{A})^c \) is open, so
\[
(\bar{A})^c \subset \text{int}(A^c)
\]
by 1'(a).

Replacing \( A \) with \( A^c \), we get \( (\bar{A}^c)^c \subset \text{int}((A^c)^c) = \text{int } A \), so
\[
(\text{int } A)^c \subset \bar{A}^c.
\]

Thus we've proved one inclusion for each of the two desired equalities. For the reverse inclusions, we have \( \text{int } A \subset A \), so \( A^c \subset (\text{int } A)^c \); because \( \text{int } A \) is open, \( (\text{int } A)^c \) is closed, so
\[
\bar{A}^c \subset (\text{int } A)^c.
\]

Replacing \( A \) with \( A^c \), we get \( \bar{A} \subset (\text{int}(A^c))^c \), so
\[
\text{int}(A^c) \subset (\bar{A})^c
\]
by 1(a).

Thus both equalities are proved.
(b) The \textit{boundary} of \(A\), denoted \(\partial A\), is defined to be \(\bar{A} \setminus \text{int} \ A\). Show that \(\partial A = \partial (X \setminus A)\).

\textbf{Solution:} We have
\[
\partial A = \bar{A} \setminus \text{int} \ A = \bar{A} \cap (\text{int} \ A)^c = \bar{A} \cap \overline{A^c},
\]
the last equality by part (a). The last expression is clearly symmetric between \(A\) and \(A^c\).

3. Find the closure, interior, and boundary of each subset of \(\mathbb{R}^2\) (in the usual topology):

(a) \(A_1 = \{ (x, y) : 0 < x \leq 1, 0 \leq y < 1 \}\)

\textbf{Solution:} The interior is the open square given by \(0 < x < 1\) and \(0 < y < 1\). The closure is the closed square \(0 \leq x \leq 1\) and \(0 \leq y \leq 1\). The boundary is the four line segments shown:

(b) \(A_2 = \{ (x, y) : 0 < x \leq 1, y = 0 \}\)

\textbf{Solution:} The interior is empty. The closure is the line segment given by \(0 \leq x \leq 1\) and \(y = 0\). The boundary is the same line segment.

(c) \(A_3 = \{ (x, y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q} \}\)

\textbf{Solution:} The interior is empty. The closure is all of \(\mathbb{R}^2\). The boundary is all of \(\mathbb{R}^2\).
4. Let \( X \) be a set, and let \( T \) be the set of subsets \( U \subset X \) that such that \( X \setminus U \) is finite, together with the empty set.

(a) Show that \( T \) is a topology. (It is called the "finite complement topology").

**Solution:** We have \( \emptyset \in T \) by definition. To see that \( X \in T \), note that \( X \setminus X = \emptyset \) is finite.

Suppose that \( U_1, \ldots, U_n \in T \). If some \( U_i = \emptyset \) then \( U_1 \cap \cdots \cap U_n = \emptyset \in T \). Otherwise \( X \setminus U_i \) is finite for all \( i \), so \( X \setminus (U_1 \cap \cdots \cap U_n) = (X \setminus U_1) \cup \cdots \cup (X \setminus U_n) \) is a finite union of finite sets, hence is finite, so \( U_1 \cap \cdots \cap U_n \in T \).

Lastly, given an arbitrary subset \( S \subset T \), we want to show that \( \bigcup S \in T \). If \( S = \emptyset \) or \( S = \{\emptyset\} \) then \( \bigcup S = \emptyset \in T \). Otherwise there is some non-empty \( U \in S \), so \( X \setminus U \) is finite. Then \( U \subset \bigcup S \), so \( X \setminus \bigcap S \subset X \setminus U \) is a subset of a finite set, hence is finite, so \( \bigcup S \in T \).

(b) Find the closure, interior, and boundary of \( \mathbb{Z} \) as a subset of \( \mathbb{R} \) in the finite complement topology.

**Solution:** I claim that the only closed set containing \( \mathbb{Z} \) is the whole set \( \mathbb{R} \). To see this, let \( F \) be a closed set containing \( \mathbb{Z} \), so \( U = \mathbb{R} \setminus F \) is an open set with \( \mathbb{Z} \) in its complement. By definition, an open set is either \( \emptyset \), or has finite complement. Since \( \mathbb{Z} \) is infinite, we must have \( U = \emptyset \), so \( F = \mathbb{R} \).

Thus \( \overline{\mathbb{Z}} = \mathbb{R} \). Similarly, \( \mathbb{R} \setminus \mathbb{Z} \) is infinite, so \( \mathbb{R} \setminus \mathbb{Z} = \mathbb{R} \), so \( \text{int} \mathbb{Z} = \emptyset \) by 3(a). Finally, \( \partial \mathbb{Z} = \overline{\mathbb{Z}} \setminus \text{int} \mathbb{Z} = \mathbb{R} \setminus \emptyset = \mathbb{R} \).
5. Let $T$ be the set of subsets $U \subset \mathbb{R}$ such that $U$ contains 0, together with the empty set.

(a) Show that $T$ is a topology.

**Solution:** We have $\emptyset \in T$ by definition, and $\mathbb{R} \in T$ because $0 \in \mathbb{R}$.

Suppose that $U_1, \ldots, U_n \in T$. If some $U_i = \emptyset$ then $U_1 \cap \cdots \cap U_n = \emptyset \in T$. Otherwise $0 \in U_i$ for all $i$, so $0 \in U_1 \cap \cdots \cap U_n$, so $U_1 \cap \cdots \cap U_n \in T$. (Indeed, we can see that $T$ is closed under arbitrary intersections, not just finite intersections.)

Lastly, given an arbitrary subset $S \subset T$, we want to show that \( \bigcup S \in T \). If $S = \emptyset$ or $S = \{\emptyset\}$ then $\bigcup S = \emptyset \in T$. Otherwise there is some non-empty $U \in S$, so $0 \in U$, so $0 \in \bigcup S$, so $\bigcup S \in T$.

(b) Find the closure, interior, and boundary of the one-point subsets \{1\} and \{0\}.

**Solution:** A set is open if and only if it either contains 0, or is empty. Thus a set is closed if and only if it either does not contain 0, or is the whole space $\mathbb{R}$.

Thus \{1\} is closed, and it contains no non-empty open set, so its interior is $\emptyset$, its closure is \{1\}, and its boundary is \{1\}, just as in the usual topology.

On the other hand, \{0\} is open, and it is not contained in any closed set apart from the whole space $\mathbb{R}$, so its interior is \{0\}, its closure is $\mathbb{R}$, and its boundary is $(-\infty,0) \cup (0,\infty)$, quite unlike the usual topology.