1. The lower semi-continuous topology on $\mathbb{R}$ has the following open sets:
$\emptyset$, all of $\mathbb{R}$, and sets of the form $(a, \infty)$ for some $a \in \mathbb{R}$.

(a) Show that the constant sequence $x_1 = x_2 = \cdots = 1$ converges to 1, and also to 0, but not to 2. That is, in the limit we can jump down, but not up.

Solution: Recall that a sequence converges to $\ell$ if for every open set $U$ with $\ell \in U$, there is an $N$ such that $n \geq N$ implies $x_n \in U$.

First we show that $x_n \to 1$. Let $U$ be an open set containing 1. The either $U = (a, \infty)$ for some $a < 1$, or $U = \mathbb{R}$. In either case $x_n \in U$ for all $n$, so we can take $N = 1$.

Next we show that $x_n \to 0$. Let $U$ be an open set containing 0. The either $U = (a, \infty)$ for some $a < 0$, or $U = \mathbb{R}$. In either case $x_n \in U$ for all $n$, so we can take $N = 1$.

Last we show that $x_n \not\to 2$. Let $U = (3/2, \infty)$, which contains 2. Then $x_n \notin U$ for any $n$, so there is no $N$ that does the job.

(b) Let $X$ be $\mathbb{R}$ with the usual topology, let $Y$ be $\mathbb{R}$ with the lower semi-continuous topology, and consider the functions $f, g: X \to Y$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x \geq 1 \\ 1 & \text{if } 0 < x < 1. \end{cases} \quad g(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 1 \\ 1 & \text{if } 0 \leq x \leq 1. \end{cases}$$

Graph both functions. Which one is continuous?
Solution: The two functions look like this:

\[
\begin{array}{c}
\text{function } f \\
\text{function } g \\
\end{array}
\]

The first function \( f \) is continuous. Let \( U \subseteq Y \) be an open set in the lower semi-continuous topology. If \( U = \mathbb{R} \), or if \( U = (a, \infty) \) for some \( a < 0 \), then \( f^{-1}(U) = \mathbb{R} \). If \( U = (a, \infty) \) for some \( 0 \leq a < 1 \), then \( f^{-1}(U) = (0, 1) \). If \( U = (a, \infty) \) for some \( a \geq 1 \), or if \( U = \emptyset \), then \( f^{-1}(U) = \emptyset \). In each case \( f^{-1}(U) \) is open in the usual topology.

The second function \( g \) is not continuous. The set \( U = (1/2, \infty) \) is open in the lower semi-continuous topology, but \( g^{-1}(U) = (-\infty, 0] \cup [1, \infty) \) is not open in the usual topology.

2. Let \( X \) and \( Y \) be topological spaces. By definition, a map \( f : X \rightarrow Y \) is continuous if for every open set \( V \subseteq Y \), the preimage \( f^{-1}(V) \) is open in \( X \). What should it mean for \( f \) to be continuous at a point \( p \in X \)?

Solution: For every open set \( V \subseteq Y \) containing \( f(p) \), there is an open set \( U \subseteq X \) with \( p \in U \subseteq f^{-1}(V) \).

Or even better: For every open set \( V \subseteq Y \) containing \( f(p) \), we have \( p \in \text{int}(f^{-1}(V)) \).