Homework 6

Due Friday, November 2, 2018

Do either 3 or 3'. This week, \( \mathbb{R} \) always has the usual topology.

1. Recall from lecture that if \( X \) is a topological space and \( Y \subset X \), the 
   \textit{subspace topology} on \( Y \) is defined by saying that a subset \( V \subset Y \) is 
   open if and only if there is an open \( U \subset X \) such that \( U \cap Y = V \). 
   Moreover, we proved that a subset \( G \subset Y \) is closed in the subspace topology if and only if there is a closed \( F \subset X \) such that \( F \cap Y = G \). 
   For the parts below, let \( A \subset Y \subset X \).

   (a) Show that if \( A \) is closed in \( Y \) and \( Y \) is closed in \( X \), then \( A \) is 
       closed in \( X \).

   (b) Give two examples to show that if \( A \) is closed in \( Y \) and \( Y \) is \textit{not} 
       closed in \( X \), then \( A \) may or may not be closed in \( X \).

   (c) Show that if \( A \) is open in \( Y \) and \( Y \) is open in \( X \), then \( A \) is open 
       in \( X \).

   (d) Give two examples to show that if \( A \) is open in \( Y \) and \( Y \) is \textit{not} 
       open in \( X \), then \( A \) may or may not be open in \( X \).

On the next midterm there will almost certainly be something about 
the closure and interior of \( A \) in \( Y \) versus in \( X \).

2. (a) Let \( X \) be a topological space, and suppose that we can write 
   \( X = F_1 \cup \ldots \cup F_n \), where each \( F_i \) is closed. Let \( Y \) be another 
   topological space, let \( f : X \to Y \), and let \( f_i : F_i \to Y \) be the 
   restriction of \( f \) to \( F_i \); that is, for \( x \in F_i \) we set \( f_i(x) = f(x) \). 
   Show that \( f \) is continuous if and only if \( f_i \) is continuous for all 
   \( i \). Hint: Use the fact that a map is continuous if and only if the 
   preimage of every closed set is closed.
(b) This is usually applied to show that a piecewise function is continuous. Consider the function \( f: [0, 1] \to \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 
0 & \text{if } x \leq 1/3, \\
3x - 1 & \text{if } 1/3 \leq x \leq 2/3, \\
1 & \text{if } x \geq 2/3.
\end{cases}
\]

If we wanted to apply part (a) to show that \( f \) is continuous, what should sets should we take for the \( F_i \)?

3. Show that the addition map \( a: \mathbb{R}^2 \to \mathbb{R} \), defined by \( a(x, y) = x + y \), is continuous. Fill in the details of the following outline.

(a) For all \( b \in \mathbb{R} \), the inverse image of the interval \((b, \infty) \subset \mathbb{R}\) is a union of (infinite) rectangles as shown:

\[
\begin{array}{c}
\begin{array}{c}
\text{rectangle 1}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{rectangle 2}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{rectangle 3}
\end{array}
\end{array}
\]

Hence \( a^{-1}((b, \infty)) \) is open, by definition of the product topology on \( \mathbb{R}^2 \).

(b) Similarly, \( a^{-1}((\infty, c)) \) is open for all \( c \in \mathbb{R} \). (Just copy this sentence and leave it at that.)

(c) If \( X \) is any topological space and \( f: X \to \mathbb{R} \) a map with the property that \( f^{-1}((b, \infty)) \) and \( f^{-1}((\infty, c)) \) are open for all \( b, c \in \mathbb{R} \), then \( f \) is continuous.
3'. Same but with the multiplication map \( m(x, y) = xy \). You will probably have to treat \( b < 0 \), \( b = 0 \), and \( b > 0 \) separately.

![Graph showing three regions: b > 0, b = 0, b < 0.]

4. In lecture we considered the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) given by

\[
f(x, y) = \begin{cases} 
\frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

We saw that it was continuous as a function of \( x \) for each fixed \( y \), and as a function of \( y \) for each fixed \( x \), but not continuous as a function of \( x \) and \( y \). Find an open set \( U \subset \mathbb{R} \) such that \( f^{-1}(U) \) is not open in \( \mathbb{R}^2 \).

Hint: To analyze \( f \) away from the origin, it may help to switch to polar coordinates, and to look up the formula for \( \sin(2\theta) \).

5. What is one question you have about last week’s lectures?