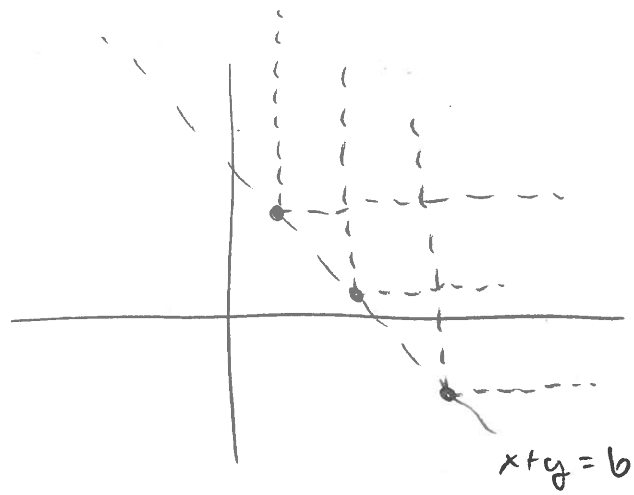
Homework 6

Due Monday, November 11, 2019

This week, $\mathbb{R}$ always has the usual topology.

1. Show that the addition map $a\colon \mathbb{R}^2 \to \mathbb{R}$, defined by $a(x, y) = x + y$, is continuous. Fill in the details of the following outline.

   (a) For all $b \in \mathbb{R}$, the inverse image of the interval $(b, \infty) \subset \mathbb{R}$ is a union of (infinite) rectangles as shown:

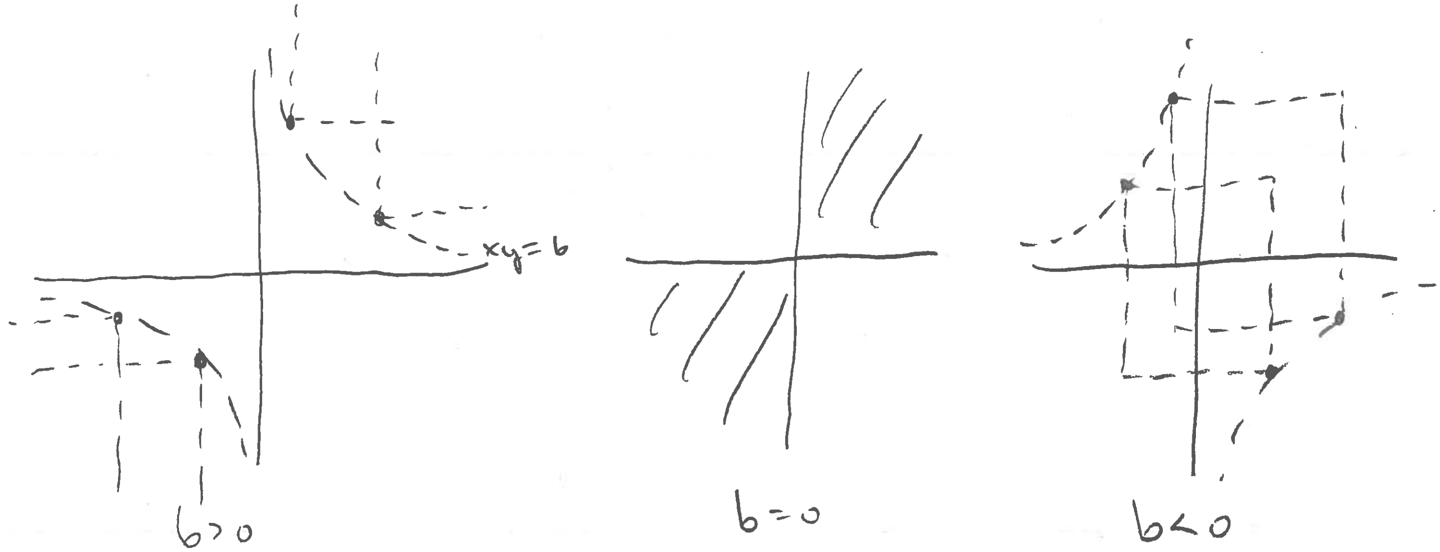
   

   Hence $a^{-1}((b, \infty))$ is open, by definition of the product topology on $\mathbb{R}^2$.

   (b) Similarly, $a^{-1}((-\infty, c))$ is open for all $c \in \mathbb{R}$. (Just copy this sentence and leave it at that.)

   (c) If X is any topological space and $f\colon X \to \mathbb{R}$ a map with the property that $f^{-1}((b, \infty))$ and $f^{-1}((-\infty, c))$ are open for all $b, c \in \mathbb{R}$, then f is continuous.

2. Optional: Same but with the multiplication map $m(x, y) = xy$. You will probably have to treat $b < 0$, $b = 0$, and $b > 0$ separately.



3. Consider the equivalence relation on $\mathbb{R}$ defined by saying that $x \sim y$ if $x = ay$ for some $a > 0$.

   (a) Show that this is an equivalence relation: reflexive, symmetric, and transitive.

   (b) Describe the equivalence classes. Hint: There are finitely many.

   (c) Describe the quotient topology on $\mathbb{R}/\sim$. What are the open sets?

4. Let $p \colon X \to Y$ be a continuous surjection.

   (a) Show that if p is open — that is, if $f(U)$ is open in Y for every open set $U \subset X$ — then p is a quotient map.

   (b) Show that if p is closed — that is, if $f(F)$ is closed in Y for every closed set $F \subset X$ — then p is a quotient map.

5. Let $X = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ or } y = 0\}$, let $Y = \mathbb{R}$, and let $p \colon X \to Y$ be the map $p(x, y) = x$. Show that p is neither open, nor closed, but that it is a quotient map.

6. What is one question you have about last week's lectures?