1. In lecture we saw that a space $X$ is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is closed. Describe the closure of the diagonal in $\mathbb{R}^2$ for one of the following non-Hausdorff topologies on $\mathbb{R}$:

(a) From homework 3 #4, the finite complement topology: $U \subset \mathbb{R}$ is open if either $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite; equivalently, $F \subset \mathbb{R}$ is closed if either $F = \mathbb{R}$ or $F$ is finite.

(b) From homework 3 #5: $U \subset \mathbb{R}$ is open if either $U = \emptyset$ or $0 \in U$; equivalently, $F \subset \mathbb{R}$ is closed if either $F = \mathbb{R}$ or $0 \notin F$.

(c) From homework 5 #1, the lower semi-continuous topology: the open sets are $\emptyset$, all of $\mathbb{R}$, and sets of the form $(-\infty, a)$ for $a \in \mathbb{R}$; equivalently, the closed sets are $\emptyset$, all of $\mathbb{R}$, and sets of the form $[a, \infty)$ for $a \in \mathbb{R}$.

2. Let $X$ and $Y$ be topological spaces, and let $f, g: X \to Y$ be two continuous maps. Show that if $Y$ is Hausdorff then the set

$$ E = \{ x \in X : f(x) = g(x) \} $$

is closed. (Hint: Think about $f \times g: X \to Y \times Y$.)

3. Let $X$ be a topological space. A subset $A \subset X$ is called dense if $\overline{A} = X$.

(a) Which of the following is dense in $\mathbb{R}$ with the usual topology? With the finite complement topology? (No proofs.)

(i) $\mathbb{Q}$. (ii) $\mathbb{R} \setminus \mathbb{Q}$. (iii) $\mathbb{Z}$. (iv) $\mathbb{R} \setminus \mathbb{Z}$.

(b) Show that $A \subset X$ is dense if and only if every non-empty open $U \subset X$ meets $A$, that is, $A \cap U \neq \emptyset$. 

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(c) Suppose that $A \subset X$ is dense, and $Y$ is Hausdorff. Show that two continuous maps $f, g : X \rightarrow Y$ that agree on $A$ must agree on all of $X$: that is, if $f(a) = g(a)$ for all $a \in A$ then $f(x) = g(x)$ for all $x \in X$. Hint: Use #2.

(d) Give a counterexample to (c) when $Y$ is not Hausdorff.

(One possibility is let $X = \mathbb{R}$ with the usual topology, and $Y = \mathbb{R}$ with one of the topologies from problem 1.)

4. Show that $X = [0, 1)$ is not homeomorphic to $Y = (0, 1)$, with the usual topologies on both.

Hint: A homeomorphism $f : X \rightarrow Y$ would induce a homeomorphism from $X \setminus \{0\}$ to $Y \setminus \{f(0)\}$.

5. Suppose that $A \subset X$ is connected. Show that $\bar{A}$ is connected.

Hint: Suppose we could write $\bar{A} = F \cup G$, where $F$ and $G$ are non-empty, closed in $\bar{A}$, and $F \cap G = \emptyset$.

6. Let $X$ be a set.

- An equivalence relation on $X$ is a relation $\sim$ that is reflexive ($x \sim x$), symmetric (if $x \sim y$ then $y \sim x$), and transitive (if $x \sim y$ and $y \sim z$ then $x \sim z$).
- A partition of $X$ is a collection of disjoint subsets of $X$ whose union is all of $X$.
- A surjection from $X$ to another set is an “onto” map.

(a) Show that an equivalence relation on $X$ determines a partition of $X$, and vice versa.

(b) Show that an equivalence relation on $X$ determines a surjection onto another set, and vice versa.

7. What is one question you have about last week’s lectures?