Homework 7

Due Monday, November 18, 2019

1. In lecture we saw that a space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is closed. Describe the closure of the diagonal in $\mathbb{R}^2$ for two of the following non-Hausdorff topologies on $\mathbb{R}$ from Homework 4. Optional: Do all three.

   (a) From problem 4, the finite complement topology: $U \subset \mathbb{R}$ is open if either $U = \varnothing$ or $\mathbb{R} \setminus U$ is finite; equivalently, $F \subset \mathbb{R}$ is closed if either $F = \mathbb{R}$ or F is finite.

   (b) From problem 5: $U \subset \mathbb{R}$ is open if either $U = \varnothing$ or $0 \in U$; equivalently, $F \subset \mathbb{R}$ is closed if either $F = \mathbb{R}$ or $0 \notin F$.

   (c) From problem 6, the lower semi-continuous topology: the open sets are $\varnothing$, all of $\mathbb{R}$, and sets of the form (a, ∞) for $a \in \mathbb{R}$; equivalently, the closed sets are $\varnothing$, all of $\mathbb{R}$, and sets of the form $(-\infty, a]$ for $a \in \mathbb{R}$.

2. Let X and Y be topological spaces, and let $f, g\colon X \to Y$ be two continuous maps. Show that if Y is Hausdorff then the set

$$E = \{x \in X : f(x) = g(x)\}$$

   is closed. (Hint: Think about $f \times g\colon X \to Y \times Y$.)

3. Let X be a topological space. A subset $A \subset X$ is called *dense* if $\bar{A} = X$.

   (a) Which of the following is dense in $\mathbb{R}$ with the usual topology? With the finite complement topology? (No proofs.)
   (i) $\mathbb{Q}$. (ii) $\mathbb{R} \setminus \mathbb{Q}$. (iii) $\mathbb{Z}$. (iv) $\mathbb{R} \setminus \mathbb{Z}$.

   (b) Show that $A \subset X$ is dense if and only if every non-empty open $U \subset X$ meets A, that is, $A \cap U \neq \varnothing$.

(c) Suppose that $A \subset X$ is dense, and Y is Hausdorff. Show that two continuous maps $f, g\colon X \to Y$ that agree on A must agree on all of X: that is, if $f(a) = g(a)$ for all $a \in A$ then $f(x) = g(x)$ for all $x \in X$. Hint: Use #2.

(d) Give a counterexample to (c) when Y is not Hausdorff.

(One possibility is to let $X = \mathbb{R}$ with the usual topology, and $Y = \mathbb{R}$ with one of the topologies from problem 1.)

4. Suppose that $A \subset X$ is connected. Show that $\bar{A}$ is connected.

   Hint: Suppose we could write $\bar{A} = F \cup G$, where F and G are non-empty, *closed* in $\bar{A}$, and $F \cap G = \varnothing$.

5. Optional: Show that $X = [0, 1)$ is not homeomorphic to $Y = (0, 1)$, with the usual topologies on both.

   Hint: A homeomorphism $f\colon X \to Y$ would induce a homeomorphism from $X \setminus \{0\}$ to $Y \setminus \{f(0)\}$.

6. What is one question you have about last week's lectures?