# Homework 8

Due Monday, November 25, 2019

1. Let $X$ be a topological space. Define a relation $\sim$ on $X$ by declaring that $p \sim q$ iff there is a path from $p$ to $q$, that is, a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

   (a) Show that this is an equivalence relation: reflexive, symmetric, and transitive. Hint: The interesting one is transitive.

   (b) The equivalence classes of this equivalence relation are called are called *path components*. Describe (without proof) the path components of the following spaces:

      i. $\{(x, y) \in \mathbb{R}^2 : xy > 1\}$.

      ii. $\mathbb{Q}$.

      iii. The topologist's sine curve.

2. Let $X$ be a topological space. Define a relation $\sim$ on $X$ by declaring that $p \sim q$ iff there is a connected subspace $A \subset X$ containing $p$ and $q$.

   (a) Show that this is an equivalence relation: reflexive, symmetric, and transitive. Hint: The interesting one is transitive.

   (b) The equivalence classes of this equivalence relation are called are called *connected components*. Describe (without proof) the connected components of the following spaces:

      i. $\{(x, y) \in \mathbb{R}^2 : xy > 1\}$.

      ii. $\mathbb{Q}$.

      iii. The topologist's sine curve.

   (c) Let $p \in X$, let $P$ be the path component of $p$, and let $C$ be the connected component of $p$. Show that $P \subset C$.

3. (a) Let $X$ be a connected space, and let $\sim$ be an equivalence relation on $X$. Show that if the equivalence classes of $\sim$ are open then every point in $X$ is equivalent to every other point.

   (b) Show that a connected open subset $U \subset \mathbb{R}^n$ is path-connected. Hint: Show that the path components of $U$ are open.

4. Optional: For the following topological spaces $X$, describe the quotient topology on $X/{\sim}$, where $\sim$ is the equivalence relation from problem 1 (not problem 2):

   (a) The topologist's sine curve.

   (b) $\mathbb{Q}$

   (c) $[\frac{1}{2}, 1] \cup [\frac{1}{4}, \frac{1}{3}] \cup [\frac{1}{6}, \frac{1}{5}] \cup \cdots$.

5. If you've seen a little algebraic topology, read this problem and think about it, but don't write it up. Let $X$ be a metric space, let $x_0 \in X$, and let $\Omega$ be the space of loops based at $x_0$, that is, the set of continuous maps $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$, endowed with the sup metric.

   (a) A path in $\Omega$ is a homotopy.

   (b) The space $X$ is semi-locally simply connected if and only if the quotient topology on the space of path components $\Omega/{\sim}$, where $\sim$ is the equivalence relation from problem 1, is discrete.

6. What is one question you have about last week's lectures?