Solutions to Homework 8

1. Let $X$ be a topological space. Define a relation $\sim$ on $X$ by declaring that $p \sim q$ iff there is a path from $p$ to $q$, that is, a continuous map $\gamma: [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

(a) Show that this is an equivalence relation: reflexive, symmetric, and transitive. Hint: The interesting one is transitive.

Solution: First we show that $\sim$ is reflexive, that is, $x \sim x$ for all $x \in X$. The constant map $\gamma: [0, 1] \to X$ defined by $\gamma(t) = x$ is a path from $x$ to $x$.\footnote{If you want to prove that $\gamma$ is continuous, let $U \subset X$ be open. If $x \in U$ then $\gamma^{-1}(U) = [0, 1]$, which is open in $[0, 1]$. If $x \notin U$ then $\gamma^{-1}(U) = \emptyset$, which is also open.}

Next we show $\sim$ is symmetric, that is, if $x \sim y$ then $y \sim x$. Let $\gamma$ be a path from $x$ to $y$. I claim that the map $\gamma': [0, 1] \to X$ defined by

$$
\gamma'(t) = \gamma(1-t)
$$

is a path from $y$ to $x$. Clearly $\gamma'(0) = \gamma(1) = y$ and $\gamma'(1) = \gamma(0) = x$. To see that $\gamma'$ is continuous, observe that the function $[0, 1] \to [0, 1]$ given by $t \mapsto 1 - t$ is continuous, and composing this with $\gamma$ gives a continuous map $\gamma'$.

Last we show that $\sim$ is transitive, that is, if $x \sim y$ and $y \sim z$ then $x \sim z$. Let $\alpha$ be a path from $x$ to $y$, and $\beta$ a path from $y$ to $z$. I claim that the map $\gamma: [0, 1] \to X$ defined by

$$
\gamma(t) = 
\begin{cases} 
\alpha(2t) & \text{if } t \in [0, \frac{1}{2}], \\
\beta(2t - 1) & \text{if } t \in \left[\frac{1}{2}, 1\right]
\end{cases}
$$

is a path from $x$ to $z$. Clearly $\gamma(0) = \alpha(0) = x$ and $\gamma(1) = \beta(1) = z$, and $\gamma$ is well-defined because at $t = \frac{1}{2}$ we have $\alpha(1) = y = \beta(1)$.\footnote{If you want to prove that $\gamma$ is continuous, let $U \subset X$ be open. If $x \in U$ then $\gamma^{-1}(U) = [0, 1]$, which is open in $[0, 1]$. If $x \notin U$ then $\gamma^{-1}(U) = \emptyset$, which is also open.}
\( \beta(0) \). To see that \( \gamma \) is continuous, we argue as follows. The map \([0, \frac{1}{2}] \to [0, 1] \) given by \( t \mapsto 2t \) is continuous, and composing this with \( \alpha \) gives a continuous map \([0, \frac{1}{2}] \to X \). Similarly, the map \([\frac{1}{2}, 1] \to [0, 1] \) given by \( t \mapsto 2t - 1 \) is continuous, and composing this with \( \beta \) gives a continuous map \([\frac{1}{2}, 1] \to X \). From this we deduce that \( \gamma \) is continuous by homework 6 #2.

(b) The equivalence classes of this equivalence relation are called are called path components. Describe (without proof) the path components of the following spaces:

i. \( \{(x, y) \in \mathbb{R}^2 : xy > 1 \} \).
   
   **Solution:** There are two path components: the half in the first quadrant, and the half in the third quadrant.

![Diagram showing two path components](image)

ii. \( \mathbb{Q} \).
   
   **Solution:** Each path component consists of a single point. The image of a path \( \gamma : [0, 1] \to \mathbb{Q} \subset \mathbb{R} \) is connected, hence is an interval in \( \mathbb{R} \), but any interval other than a single point contains an irrational number. Thus every path in \( \mathbb{Q} \) is constant.

iii. The topologist's sine curve.
   
   **Solution:** There are two path components: the set
   
   \[ A = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = 1/x \}, \]
   
   and the line segment \( \{0\} \times [-1, 1] = \overline{A} \setminus A \).
2. Let $X$ be a topological space. Define a relation $\sim$ on $X$ by declaring that $p \sim q$ iff there is a connected subspace $A \subset X$ containing $p$ and $q$.

(a) Show that this is an equivalence relation: reflexive, symmetric, and transitive. Hint: The interesting one is transitive.

**Solution:**

Reflexive: Given $x \in X$, we take $A = \{x\}$, which is connected.

Symmetric: If $A \subset X$ is connected and contains both $x$ and $y$, then it contains both $y$ and $x$.

Transitive: If $A \subset X$ is connected and contains $x$ and $y$, and if $B \subset X$ is connected and contains $y$ and $z$, then $A \cup B$ contains $x$ and $z$, and I claim that it is connected.

Let $2 = \{0, 1\}$ with the discrete topology, and recall that a space is connected if and only if every continuous map to 2 is constant. Let $f : A \cup B \to 2$. Because $A$ and $B$ are connected, the restrictions $f|_A$ and $f|_B$ are constant. Because $y \in A \cap B$, we see that $f|_A$ and $f|_B$ take the same value $f(y)$ throughout $A \cup B$.

(b) The equivalence classes of this equivalence relation are called are called connected components. Describe (without proof) the connected components of the following spaces:

i. $\{(x, y) \in \mathbb{R}^2 : xy > 1\}$.

**Solution:** Same as the path components: one in the first quadrant, and one in the third quadrant.

ii. $\mathbb{Q}$.

**Solution:** Same as the path components: single points.

iii. The topologist’s sine curve.

**Solution:** Because $\bar{A}$ is connected, it has only one connected component.

(c) Let $p \in X$, let $P$ be the path component of $p$, and let $C$ be the connected component of $p$. Show that $P \subset C$.

**Solution:** Let $q \in P$, so there is a continuous $\gamma : [0, 1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$. Now $[0, 1]$ is connected, and the continuous image of a connected set is connected, so $\gamma([0, 1])$ is a connected subset of $X$ containing both $p$ and $q$, so $q \in C$. 

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3. (a) Let \( X \) be a connected space, and let \( \sim \) be an equivalence relation on \( X \). Show that if the equivalence classes of \( \sim \) are open then every point in \( X \) is equivalent to every other point.

**Solution:** Fix some \( x \in X \), and let \( U \) be the equivalence class of \( x \), which is open by hypothesis. Let \( V \) be the union of all the other equivalence classes, which is a union of open sets, hence is open. Then we have \( X = U \cup V \) and \( U \cap V = \emptyset \). Because \( X \) is connected, it must be that \( U = X \) and \( V = \emptyset \).

(b) Show that a connected open subset \( U \subset \mathbb{R}^n \) is path-connected. Hint: Show that the path components of \( U \) are open.

**Solution:** Following the hint, let \( P \subset U \) be a path component; we want to show that \( P \) is open. Let \( p \in P \). Because \( U \) is open, we can choose an \( \epsilon > 0 \) such that \( B_\epsilon(p) \subset U \). I claim that \( B_\epsilon(p) \subset P \): for any \( q \in B_\epsilon(p) \), the straight line from \( p \) to \( q \) is a path that stays in \( B_\epsilon(p) \), hence in \( U \), so \( q \) is in the same path component as \( p \).

Because the path components of \( U \) are open, by part (a) we know that there is only one path component, that is, \( U \) is connected.

4. Optional: For the following topological spaces \( X \), describe the quotient topology on \( X/\sim \), where \( \sim \) is the equivalence relation from problem 1 (not problem 2):

(a) The topologist’s sine curve.

**Solution:** Sierpinsky space: There are two points, one open but not closed and the other closed but not open.

(b) \( \mathbb{Q} \)

**Solution:** It’s the same as the original space \( \mathbb{Q} \): since the path components are single points, the quotient map is a bijection, so it is a homeomorphism.

(c) \( [\frac{1}{2}, 1] \cup [\frac{1}{3}, \frac{1}{2}] \cup [\frac{1}{5}, \frac{1}{3}] \cup \ldots \).

**Solution:** Each path component is open, so each point of the quotient is open, so the quotient is a countable discrete space.