1. Which of the following topologies on $\mathbb{R}$ are compact? Give a proof either way.
   (a) The finite complement topology.
   (b) The topology where $U$ is open iff either $U = \emptyset$ or $0 \in U$.
   (c) The lower semi-continuous topology.

2. (a) Let $X$ be a compact space, and let $F_1 \supset F_2 \supset F_3 \supset \cdots$ be a descending chain of non-empty closed subsets. Show that the intersection $F_1 \cap F_2 \cap F_3 \cap \cdots$ is not empty.
   Hint: Otherwise $X \setminus F_1$, $X \setminus F_2$, $\ldots$ is an open cover of $X$.
   (b) Give an example of a non-compact space $X$ and a descending chain of closed subsets $F_1 \supset F_2 \supset F_3 \supset \cdots$ whose intersection is empty.

3. (a) Let $X$ and $Y$ be topological spaces, let $A \subset X$, and let $U$ be a neighborhood of $A \times Y$ in $X \times Y$. Show that if $Y$ is compact then there is a neighborhood $V$ of $A$ in $X$ such that $V \times Y \subset U$.
   (Start by drawing a picture!)
   (b) Give a counterexample when $Y$ is not compact.

4. A continuous map $f : X \to Y$ is called proper if the preimage of any compact set $K \subset Y$ is compact.
   (a) Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is proper.
   (b) Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 - y^2$ is not proper.
(c) If \( f \) is proper then the preimage of every point is compact, because points are compact. But give an example of a continuous map \( f: X \to Y \) for which the preimage of every point is compact, but nonetheless \( f \) is not proper.

(d) Show that if \( X \) is compact and \( Y \) is Hausdorff then any continuous map \( f: X \to Y \) is proper.

(e) Let \( X \) and \( Y \) be topological spaces. Show that the projection \( p: X \times Y \to X \) is proper if and only if \( Y \) is compact.

5. Optional. We define one-point compactification of a topological space \( X \) to be \( \hat{X} = X \cup \{\infty\} \), with the following topology: if \( U \) is open in \( X \), then \( U \) is open in \( \hat{X} \); and if \( K \subset X \) is compact and closed, then \((X \setminus K) \cup \{\infty\}\) is open in \( \hat{X} \). Equivalently, \( F \subset \hat{X} \) is closed if \( F \cap X \) is closed, and either (1) \( \infty \in F \) or (2) \( F \) is compact.

(a) Show that the one-point compactification of \([0, 1)\) is homeomorphic to \([0, 1]\), and that the one-point compactification of \( \mathbb{R} \) or \((0, 1)\) is homeomorphic to the circle. Describe the one-point compactification of \( \mathbb{R}^2 \).

(b) Show that \( \hat{X} \) is compact.

(c) A space \( X \) is called locally compact if for every point \( p \in X \) there is a compact set \( K \subset X \) with \( p \in \text{int}(K) \). This is a slightly unusual use of the word “locally,” but it the appropriate one for compactness. For example, \( \mathbb{R} \) is locally compact, but \( \mathbb{Q} \) is not. Show that \( \hat{X} \) is Hausdorff if and only if \( X \) is locally compact and Hausdorff.

(d) A map \( f: X \to Y \) induces a map \( \hat{f}: \hat{X} \to \hat{Y} \). Show that \( \hat{f} \) is continuous if and only if \( f \) is proper.