1. Let $X$ be a topological space. Define a relation $\sim$ on $X$ by declaring that $p \sim q$ iff there is a connected subspace $A \subset X$ containing $p$ and $q$.

   (a) Show that this is an equivalence relation: reflexive, symmetric, and transitive. Hint: The interesting one is transitive.

   (b) The equivalence classes of this equivalence relation are called connected components. Describe (without proof) the connected components of the following spaces:

   i. $\{(x, y) \in \mathbb{R}^2 : xy > 1\}$.

   ii. $\mathbb{Q}$.

   iii. The topologist’s sine curve.

   (c) Let $p \in X$, let $P$ be the path component of $p$, and let $C$ be the connected component of $p$. Show that $P \subset C$.

2. (a) Let $X$ be a connected space, and let $\sim$ be an equivalence relation on $X$. Show that if the equivalence classes of $\sim$ are open then every point in $X$ is equivalent to every other point.

   (b) Show that a connected open subset $U \subset \mathbb{R}^n$ is path-connected. Hint: Show that the path components of $U$ are open.

3. Which of the following topologies on $\mathbb{R}$ are compact? Give a proof either way.

   (a) The finite complement topology.

   (b) The topology where $U$ is open iff either $U = \emptyset$ or $0 \in U$.

   (c) The lower semi-continuous topology.
4. A continuous map \( f : X \to Y \) is called \textit{proper} if the preimage of any compact set \( K \subset Y \) is compact.

(a) Show that the map \( f : \mathbb{R}^2 \to \mathbb{R} \) given by \( f(x, y) = x^2 + y^2 \) is proper.

(b) Show that the map \( f : \mathbb{R}^2 \to \mathbb{R} \) given by \( f(x, y) = x^2 - y^2 \) is not proper.

(c) Show that if \( f \) is proper then the preimage of every point is compact.

(d) Give an example of a continuous map \( f : X \to Y \) for which the preimage of every point is compact, but nonetheless \( f \) is not proper.

(e) Show that if \( X \) is compact and \( Y \) is Hausdorff then any continuous map \( f : X \to Y \) is proper.

(f) Let \( X \) and \( Y \) be topological spaces. Show that the projection \( p : X \times Y \to X \) is proper if and only if \( Y \) is compact.

5. What is one question you have about last week’s lectures?