Here is a streamlined account of the proof we came up with in class Monday. If I had it to do over again I'd pick slightly different notation.

**Definition.** A topological space $X$ is disconnected if we can write $X = U \cup V$, where $U$ and $V$ are open and non-empty, and $U \cap V = \emptyset$. The space $X$ is connected if it is not disconnected.

**Definition.** A subset $A \subset \mathbb{R}$ is an interval if for all $a, b, c \in \mathbb{R}$ with $a < b < c$, if $a \in A$ and $c \in A$ then $b \in A$.

**Theorem.** A subset $A \subset \mathbb{R}$ is connected if and only if it is an interval.

**Proof.** First let us show that if $A$ is not an interval then $A$ is disconnected. Let $a < b < c$ with $a, c \in A$ and $b \notin A$. Let $U = (-\infty, b) \cap A$ and $V = (b, \infty) \cap A$. Then $U$ and $V$ are non-empty, disjoint, open in $A$, and $A = U \cup V$.

Conversely, let us show that if $A$ is disconnected then $A$ is not an interval. Write $A = U \cup V$, where $U$ and $V$ are non-empty, disjoint, and open in $A$. Observe that $U$ and $V$ are also closed in $A$, because their complements $A \setminus U = V$ and $A \setminus V = U$ are open in $A$.

Let $a \in U$ and $c \in V$. Assume that $a < c$; if $a > c$ then the argument is similar. We want to produce some $b \notin A$ with $a < b < c$.

Let $b = \sup \{u \in U : u < c\}$. We have $a \leq b \leq c$. If $b \notin A$ then we are done. So assume that $b \in A$; then $b \in U$, because $b$ is in the closure of $U$ and $U$ is closed in $A$.

Let $b' = \inf \{v \in V : b < v\}$. We have $a \leq b \leq b' \leq c$. If $b' \notin A$ then we are done. So assume that $b' \in A$; then $b' \in V$, because $b'$ is in the closure of $V$ and $V$ is closed in $A$.

Because $b \in U$ and $b' \in V$, we have $b \neq b'$, so $b < b'$, so we can consider $b'' = \frac{b + b'}{2}$, which satisfies $a \leq b < b'' < b' \leq c$.

To finish the proof, we show that $b'' \notin A$: if $b''$ were in $U$ then we would have $b'' \leq b$, by construction of $b$ as the sup of $\{u \in U : u < c\}$; and if $b''$ were in $V$ then we would have $b' \leq b''$, by construction of $b'$ as the inf of $\{v \in V : b < v\}$.

\qed