Midterm 1

Friday, November 1, 2019

Use your own notebook paper, or get some from me. Put each problem on its own page. Each part is worth 5 points, for a total of 40 points. Assume that $\mathbb{R}$ has the usual topology unless it says otherwise.

1. (a) Let (X, d) be a metric space. Prove that all $p, q, a \in X$, we have

$$|d(p, a) - d(q, a)| \leq d(p, q).$$

This is called the "reverse triangle inequality."

(b) Let (X, d_X) and (Y, d_Y) be metric spaces. Define what it means for a map $f\colon X \to Y$ to be continuous. (Use δ and ϵ, not open sets.)

(c) Let (X, d) be a metric space, and fix a point $a \in X$. Show that the map $f\colon X \to \mathbb{R}$ given by $f(p) = d(p, a)$ is continuous.

2. (a) Let X and Y be topological spaces. Define what it means for a map $f\colon X \to Y$ to be continuous.

(b) In the *lower limit topology* on $\mathbb{R}$, the open sets are $\varnothing$ and unions of intervals of the form $[a, b)$. Note that this is different from the *lower semi-continuous topology* which appeared in lecture and on the homework.

Consider the maps $f, g\colon \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0, \end{cases} \qquad g(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0. \end{cases}$$

If the domain $\mathbb{R}$ is given the lower limit topology and the codomain (target) $\mathbb{R}$ is given the usual topology, one of f and g is continuous, and one is not. Which one is not continuous? Give a reason but not necessarily a complete proof.

3. Do either (a) or (a′). You may use any statements about images, preimages, interiors, and closures that were on the homework.

   (a) Suppose that $f\colon X \to Y$ is continuous. Show that for every subset $A \subset X$ we have $f(\bar{A}) \subset \overline{f(A)}$.

   Hint: Use the characterization of continuity in terms of preimages of *closed* sets rather than open sets.

   (a′) Suppose that $f\colon X \to Y$ is continuous. Show that for every subset $B \subset Y$ we have $f^{-1}(\operatorname{int} B) \subset \operatorname{int} f^{-1}(B)$.

   (b) Consider the discontinuous map $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

   Find a subset $A \subset \mathbb{R}$ such that $f(\bar{A}) \not\subset \overline{f(A)}$.

   (c) Consider the continuous map $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Find a subset $B \subset \mathbb{R}$ such that $f^{-1}(\operatorname{int} B) \subsetneq \operatorname{int} f^{-1}(B)$.