Midterm 2

Friday, November 22, 2019

Each part is worth 5 points, for a total of 55 points.

Tip: Read all the questions, and do the easy ones first.

1. (a) Define what it means for a topological space X to be *Hausdorff*.

   (b) For a point p in a topological space X, let A_p denote the intersection of all open sets containing p:

   $$A_p = \bigcap \{U \subset X : U \text{ is open and } p \in U\}.$$

   Show that if X is Hausdorff then $A_p = \{p\}$.

   (c) We have seen that the finite complement topology on $\mathbb{R}$ is not Hausdorff. Is it true that $A_p = \{p\}$ for every $p \in \mathbb{R}$? Give a proof or a counterexample.

   Recall that $U \subset \mathbb{R}$ is open in the finite complement topology if either $U = \varnothing$ or $\mathbb{R} \setminus U$ is finite.

   (d) Same question with the "containing zero" topology, where $U \subset \mathbb{R}$ is open if either $U = \varnothing$ or $0 \in U$.

   (e) Same question with the lower semi-continuous topology, where $U \subset \mathbb{R}$ is open if $U = \varnothing$, or $U = \mathbb{R}$, or $U = (a, \infty)$ for some $a \in \mathbb{R}$.

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2. (a) Let X be a topological space and $A \subset X$. Define the *subspace topology* on A.

   (b) Let X and Y be topological spaces. Define the *product topology* on $X \times Y$.

   (c) Let $f\colon X \to Y$ be a continuous map, and consider the *graph* of f,

$$\Gamma = \{(x, y) \in X \times Y : y = f(x)\}$$

   as a subspace of $X \times Y$. Show that Γ is homeomorphic to X.

   (d) Give an example of a discontinuous map $f\colon \mathbb{R} \to \mathbb{R}$, where both copies of $\mathbb{R}$ have the usual topology, such that the graph of f is not homeomorphic to $\mathbb{R}$.

3. (a) Let X be a topological space, let $\sim$ be an equivalence relation on X, and let $\pi\colon X \to X/{\sim}$ be the natural map to the set of equivalence classes. Define the *quotient topology* on $X/{\sim}$.

   (b) Let $X = \mathbb{R}$ with the usual topology, let $A = (-1, 1)$, and consider the space X/A obtained by collapsing A to a point: that is, we define an equivalence relation $\sim$ on X by saying that $x \sim y$ if either $x = y$ or $x, y \in A$, and then we define $X/A = X/{\sim}$ with the quotient topology.

   Show that X/A is not Hausdorff.