Final Exam

Due Friday, March 20, 2020

This is essentially §1.8 #11 with more scaffolding. You may take as much time as you need, and you may refer to Chapter 1 of Guillemin and Pollack, but you may not discuss the exam with anyone else. If you're unsure of the definitions, refer to the book.

For a point $a \in S^{n-1} \subset \mathbb{R}^n$, we considered the projection

$$p\colon \mathbb{R}^n \to a^\perp \cong \mathbb{R}^{n-1}$$

given by

$$p(w) = w - (w \cdot a)\, a.$$

We argued in lecture that because p is linear, we have $dp_w = p$ for all $w \in \mathbb{R}^n$, and that $p(w) = 0$ if and only if w is a multiple of a. You may use all this without repeating the proof.

1. Let Y and Z be manifolds of the same dimension m, and let $g\colon Y \to Z$ be a smooth map. Show that if Y is compact and z is a regular value of g then $g^{-1}(z)$ is a finite set of points.

2. Let $X \subset \mathbb{R}^N$ be a compact, k-dimensional manifold, and let $S(X) \subset T(X)$ be the sphere bundle as in §1.8 #9.

   (a) Argue that $S(X)$ is compact.
   Hint: It's a subspace of $X \times S^{N-1}$.

   (b) Let $f\colon X \to \mathbb{R}^n$ be an immersion.
   Argue that the map $F\colon S(X) \to S^{n-1}$ given by

   $$F(x, v) = \frac{df_x(v)}{|df_x(v)|}$$

   is well-defined.

   (c) For a point $a \in S^{n-1}$ and the projection p discussed above, show that $p \circ f$ fails to be an immersion at a point $x \in X$ if and only if there is a unit vector $v \in T_x(X)$ with $F(x, v) = a$.

3. Show that a compact, k-dimensional manifold admits a map to $\mathbb{R}^{2k-1}$ that is an immersion except perhaps at finitely many points.

   Hint: Start with the inclusion $f\colon X \to \mathbb{R}^N$, let a be a regular value of F, apply p, and repeat. Think about why $2k - 1$ is the place to stop.