Solutions to Homework 2

5. *Prove that $\mathbb{R}^k$ and $\mathbb{R}^l$ are not diffeomorphic if $k \neq l$.*

   Suppose that $f\colon \mathbb{R}^k \to \mathbb{R}^l$ is a diffeomorphism: so f is a smooth bijection, and f^{-1} is smooth. We have $f^{-1} \circ f = \mathrm{id}_{\mathbb{R}^k}$, so taking derivatives at 0 and applying the chain rule we get $d(f^{-1})_{f(0)} \circ df_0 = \mathrm{id}_{\mathbb{R}^k}$, and similarly $df_0 \circ d(f^{-1})_{f(0)} = \mathrm{id}_{\mathbb{R}^l}$. Thus $df_0\colon \mathbb{R}^k \to \mathbb{R}^l$ is an invertible linear map, with inverse given by $d(f^{-1})_{f(0)}$, so $\mathbb{R}^k$ and $\mathbb{R}^l$ are isomorphic as vector spaces, so $k = l$.

7. *Exhibit a basis for $T_p(S^2)$ for an arbitrary point $p = (a, b, c)$.*

   If $c > 0$ then we can parametrize the northern hemisphere by

   $$\phi(x, y) = (x, y, \sqrt{1 - x^2 - y^2}),$$

   which is smooth on the open set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. The derivative of this parametrization is

   $$d\phi_{(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1-x^2-y^2}} & \frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix},$$

   so at the point $p = \phi(a, b)$ that we're interested in, we have

   $$d\phi_{(a,b)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -a/c & -b/c \end{pmatrix},$$

   so $(1, 0, -a/c)$ and $(0, 1, -b/c)$ form a basis for T_pS^2.

   If $c < 0$ then we find that $(1, 0, -a/c)$ and $(0, 1, -b/c)$ again form a basis for T_pS^2, by a similar computation.

   If $b \neq 0$, we find that $(1, -a/b, 0)$ and $(0, -c/b, 1)$ form a basis for T_pS^2.

   If $a \neq 0$, we find that $(-b/a, 1, 0)$ and $(-c/a, 0, 1)$ form a basis for T_pS^2.

8. *What is the tangent space to the paraboloid* defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$, where $a > 0$?*

Parametrize half of the surface by

$$\phi(y, z) = (\sqrt{z^2 - y^2 + a}, y, z),$$

which is smooth on the open set $\{(y, z) \in \mathbb{R}^2 : y^2 - z^2 < a\}$. The derivative of this parametrization is

$$d\phi_{(x,y)} = \begin{pmatrix} \frac{-y}{\sqrt{z^2-y^2+a}} & \frac{z}{\sqrt{z^2-y^2+a}} \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so at the point $(\sqrt{a}, 0, 0) = \phi(0, 0)$ that we're interested in, we have

$$d\phi_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so the tangent space is the span of the vectors $(0, 1, 0)$ and $(0, 0, 1)$ in $\mathbb{R}^3$: it's the yz-plane.

*This is a mistake in the book: it's actually a hyperboloid of one sheet.

12. *A* curve *in a manifold x is a smooth map $t \mapsto c(t)$ of an inverval of $\mathbb{R}^1$ into X. The* velocity vector *of the curve c at time t_0—denoted simply $dc/dt(t_0)$—is defined to be the vector $dc_{t_0}(1) \in T_{x_0}(X)$, where $x_0 = c(t_0)$ and $dc_{t_0} \colon \mathbb{R}^1 \to T_{x_0}(X)$. In case $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \ldots, c_k(t))$ in coordinates, check that*

$$\frac{dc}{dt}(t_0) = (c_1'(t_0), \ldots, c_k'(t_0)).$$

Prove that every vector in $T_x(X)$ is the velocity vector of some curve in X, and conversely.

For the claim about $\mathbb{R}^k$, there is not much to say. From the second paragraph of page 8 we know that the derivative $dc_{t_0} \colon \mathbb{R}^1 \to \mathbb{R}^k$ is represented by the matrix

$$\begin{pmatrix} c_1'(t_0) \\ \vdots \\ c_k'(t_0) \end{pmatrix},$$

and applying this to $1 \in \mathbb{R}^1$ we get the claimed vector in $\mathbb{R}^k$.

Now let X be an arbitrary k-dimensional manifold, let $x \in X$ be a point, and let $v \in T_x(X)$ be a tangent vector at x. Choose a parametrization $\phi \colon U \to X$, where U is an open set in $\mathbb{R}^k$ and $x = \phi(u)$ for some $u \in U$. Then v is in the image of $d\phi_u$, so there is a $w \in \mathbb{R}^k$ such that $d\phi_u(w) = v$. Let

$$c(t) = \phi(u + tw),$$

which is defined for t in a neighborhood of 0. Then by the chain rule we have $dc_0(h) = d\phi_u(hw)$, so $dc/dt(0) = dc_0(1) = d\phi_u(w) = v$.

Conversely, the velocity vector of a curve c passing through $x \in X$ is an element of $T_x(X)$ by definition.