Solutions to Homework 3

5. *Prove that a local diffeomorphism $f\colon X \to Y$ is actually a diffeomorphism of X onto an open subset of Y, provided that f is one-to-one.*

    Because f is injective, it is a bijection onto its image: $X \to f(X)$. We need to show that $f(X)$ is open, and that the inverse map $f(X) \to X$ is smooth.

    First we argue that $f(X)$ is open. Because f is a local diffeomorphism, for each $x \in X$ there is an open set $U_x \subset X$ with $x \in U_x$ and an open set $V_x \subset Y$ with $f(x) \in V$, such that f restricted to U_x is a diffeomorphism $U_x \to V_x$. Thus

    $$f(X) = f(\bigcup_{x\in X} U_x) = \bigcup_{x\in X} f(U_X) = \bigcup_{x\in X} V_x$$

    is a union of open sets, hence is open. (More generally we could have shown that f is an open map.)

    Next we argue that $f^{-1}\colon f(X) \to X$ is smooth. Let $y \in f(X)$, write $y = f(x)$, and let U_x and V_x be as above. Then $f|_{U_x}\colon U_x \to V_x$ is a diffeomorphism, so $f^{-1}|_{V_x}\colon V_x \to U_x$ is smooth, and in particular f^{-1} is smooth in a neighborhood of y. Since y was arbitrary, f^{-1} is smooth on all of $f(X)$.

8. *Check that the map*

$$f : \mathbb{R} \to \mathbb{R}^2 \qquad\qquad f(t) = \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right)$$

is an embedding. Prove that its image is one [branch] of the hyperbola $x^2 - y^2 = 1$.

First we check that f is an immersion. Its derivative is

$$f'(t) = \left(\frac{e^t - e^{-t}}{2}, \frac{e^t + e^{-t}}{2}\right),$$

of which the y-component is always positive, so f' is never zero.

Before checking that f is injective and proper, we observe that the smooth map

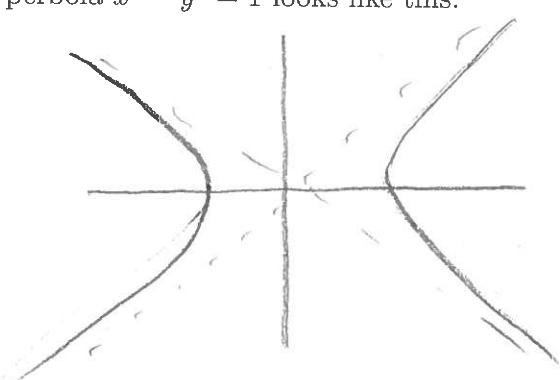
$$g : \mathbb{R}^2 \to \mathbb{R} \qquad\qquad g(x, y) = \log(y + \sqrt{y^2 + 1})$$

satisfies $g(f(t)) = t$. Thus f is injective: if $f(t_1) = f(t_2)$ then

$$t_1 = g(f(t_1)) = g(f(t_2)) = t_2.$$

And f is proper: if $K \subset \mathbb{R}^2$ is compact, then $g(K)$ is compact, so $f^{-1}(K) \subset g(K)$ is a closed subset of a compact set, hence is compact.[1] Thus f is an embedding.

The hyperbola $x^2 - y^2 = 1$ looks like this:



[1]If you want to spell out the details of $f^{-1}(K) \subset g(K)$, let $t \in f^{-1}(K)$, so $f(t) \in K$; then $t = g(f(t)) \in g(K)$.

First we check that the image of f is contained in the parabola: we have

$$\left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{e^{2t}+2+e^{-2t}}{4} - \frac{e^{2t}-2+e^{-2t}}{2} = \frac{4}{4} = 1.$$

Moreover it is contained in the right half: the x-component of f is the average of two positive numbers e^t and e^{-t}, hence is positive.

To show that the left half of the parabola is contained in the image of f, we calculate that for any $y \in \mathbb{R}$ we have

$$f(\log(y+\sqrt{y^2+1})) = (\sqrt{y^2+1}, y),$$

so every (x, y) with $x^2 - y^2 = 1$ and $x > 0$ is in the image of f.

10. *Let $f\colon X \to Y$ be a smooth map that is one-to-one on a compact subset $Z \subset X$. Suppose that for all $x \in Z$,*

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

is an isomorphism. Then f maps Z diffeomorphically onto $f(Z)$. (Why?) Prove that f, in fact, maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y.

For the first question (why?), note that $f|_Z$ is an embedding: is an immersion, because its derivative at a point $z \in Z$ is just df_z restricted to $T_z(Z) \subset T_z(X)$; and it is injective, by hypothesis; and it is proper, because Z is compact. Now quote the theorem on page 17 to say that $f|_Z$ is a diffeomorphism $Z \to f(Z)$.

Now we turn to the main question. Let $U \subset X$ be the open subset on which f is a local diffeomorphism. This is open, as we argued in class: locally, we can choose coordinates and write df as a matrix of continuous functions, so the set where $\det(df) \neq 0$ is open. Now by problem 5 it is enough to produce a smaller open set $U' \subset U$, still containing Z, on which f is injective.

We observe that f is locally injective on U: for every $x \in U$, there is an open set $U_x \subset U$ with $x \in U$ and $V_x \subset Y$ with $f(x) \in V$ such that $f|_{U_x}$ is a diffeomorphism $U_x \to V_x$, and in particular $f|_{U_x}$ is injective.

Now one approach is to follow the hint and use sequences. Let U_n be the "open tube of radius $1/n$ around Z in U," that is, $U_n = \{x \in U : \inf_{z \in Z} d(x,z) < 1/n\}$, where the metric d comes from the ambient Euclidean space in which X is embedded. If f is injective on some U_n then we win. If not, then for every n there are $a_n, b_n \in U_n$ with $a_n \neq b_n$ but $f(a_n) = f(b_n)$. Both sequences live in the set $\{x \in \mathbb{R}^n : \inf_{z \in Z} d(x,z) \leq 1\}$, which is closed and bounded because Z is. Thus we can extract a convergent subsequence of $a_{i_1}, a_{i_2}, \ldots$ of the a_is, with some limit $z \in U_1 \cap U_2 \cap \cdots = Z$. And from the subsequence $b_{i_1}, b_{i_2}, \ldots$ we can extract a convergent subsubsequence $b_{i_{j_1}}, b_{i_{j_2}}, \ldots$, with some limit $z' \in Z$. And the subsubsequence $a_{i_{j_1}}, a_{i_{j_2}}, \ldots$ still converges to z. For convenience, relabel both subsubsequences as $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$. Then $f(z) = \lim f(a_n) = \lim f(b_n) = f(z')$, and because f is injective on Z this implies $z = z'$. But f is locally injective near z, and so the two sequences eventually come into a neighborhood of z on which f is injective, contradicting the fact that $a_n \neq b_n$ and $f(a_n) = f(b_n)$.

Another approach is to prove a purely topological fact: if $f \colon U \to Y$ is continuous and locally injective, and actually injective on a compact subset $Z \subset U$, and Y is Hausdorff, then there is a neighborhood U' of Z in U on which f is injective. First, for every pair of distinct points $p, q \in Z$ we can choose neighborhoods $A_{p,q}$ of p in U and $B_{p,q}$ of q in U such that $f(A_{p,q}) \cap f(B_{p,q}) = \varnothing$: just take f^{-1} of disjoint open neighborhoods of $f(p)$ and $f(q)$, which are distinct because f is injective on Z. Next, for every $p \in Z$ choose a neighborhood W_p of p in X on which f is injective. Then for a fixed p, consider the open cover $\{W_p\} \cup \{B_{p,q} : q \in Z\}$ of Z, and extract a finite subcover $W_p, B_{p,q_1}, \ldots, B_{p,q_{n_p}}$. Let $U_p = W_p \cap A_{p,q_1} \cap \cdots \cap A_{p,q_{n_p}}$ and $V_p = W_p \cup B_{p,q_1} \cup \cdots \cup B_{p,q_{n_p}} \supset Z$. Then for $x_1 \in U_p$ and $x_2 \in V_p$ we see that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Now the U_p's cover Z, so extract a finite subcover $U_{p_1}, \ldots, U_{p_m}$, and set $U' = (U_{p_1} \cup \cdots \cup U_{p_m}) \cap (V_{p_1} \cap \cdots \cap V_{p_m})$; then U' contains Z and f is injective on U'.