Solutions to Homework 4

1. *If $f\colon X \to Y$ is a submersion and U is an open set of X, show that $f(U)$ is open in Y.*

   The point is that the projection $\mathbb{R}^k \to \mathbb{R}^l$, where $k \geq l$, is an open map, and if a map is "locally open" then it is open.

   Let $k = \dim X$ and $l = \dim Y$. By the local submersion theorem, for each $x \in X$ there is a parametrization $\phi_x\colon U_x \to X$, where $U_x \subset \mathbb{R}^k$ and $\phi_x(0) = x$, and a parametrization $\psi_x\colon V_x \to Y$, where $V_x \subset \mathbb{R}^l$ and $\psi_x(0) = f(x)$, such that $f(\phi_x(U_x)) \subset \psi_x(V_x)$ and

   $$(\psi_x^{-1} \circ f \circ \phi_x)(x_1, \ldots, x_k) = (x_1, \ldots, x_l).$$

   First observe this is an open map: an open set in $U_x \subset \mathbb{R}^k$ is a union of open boxes $(a_1, b_1) \times \cdots \times (a_k, b_k)$, so its image in $V_x \subset \mathbb{R}^l$ is a union of open boxes $(a_1, b_1) \times \cdots \times (a_l, b_l)$, hence is open.

   Next observe that

   $$f \circ \phi_x\colon U_x \to Y$$

   is an open map, because it is a composition of two open maps ψ_x and $\psi_x^{-1} \circ f \circ \phi_x$.

   Now suppose that $U \subset X$ is open. The open sets $\phi_x(U_x)$ cover X, so

   $$U = \bigcup_{x \in X} U \cap \phi_x(U_x) = \bigcup_{x \in X} \phi_x(\phi_x^{-1}(U)),$$

   so

   $$f(U) = \bigcup_{x \in X} f(\phi_x(\phi_x^{-1}(U)))$$

   is a union of open sets, hence is open.

12. *Prove that the set of all 2×2 matrices of rank 1 is a 3-dimensional submanifold of $\mathbb{R}^4 = M(2)$.*

Recall that a 2×2 matrix A has rank 1 if and only if $\det A = 0$ and $A \neq 0$. We compute the critical points of the determinant map $M(2) \to \mathbb{R}$, given by

$$\det \begin{pmatrix} x & y \\ z & w \end{pmatrix} = xw - yz.$$

Its derivative is

$$d\det_{(x,y,z,w)} = \begin{pmatrix} w & -z & -y & x \end{pmatrix},$$

which vanishes only at the origin. Thus the restriction of det to $M(2)\backslash 0$ is a submersion, so the preimage of 0, which is the set we want, is a submanifold of dimension $4 - 1 = 3$.

2. *Which of the following linear spaces intersect transversely?*

Looking at problem 1, we see that two subspaces are transverse if and only if they span the ambient space.

(a) The xy-plane and the z-axis in $\mathbb{R}^3$ are transverse.

(b) The xy-plane and the plane spanned by $(3, 2, 0)$ and $(0, 4, -1)$ in $\mathbb{R}^3$ are transverse.

(c) The plane spanned by $(1, 0, 0)$ and $(2, 1, 0)$ and the y-axis in $\mathbb{R}^3$ are not transverse: the plane contains the line.

(d) $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$: we must interpret the first 0 as being in $\mathbb{R}^{n-k}$ and the second as being in $\mathbb{R}^{n-l}$. Then the two are transverse if and only if $k + l \geq n$.

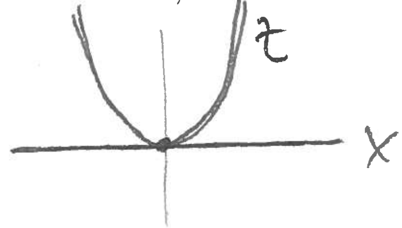
(e) $\mathbb{R}^k \times \{0\}$ and $\mathbb{R}^l \times \{0\}$ in $\mathbb{R}^n$ are transverse if and only if $k = n$ or $l = n$.

(f) $V \times \{0\}$ and the diagonal in $V \times V$ are transverse: let $(v, w) \in V \times V$ be given; then $(v, w) = (v - w, 0) + (w, w)$.

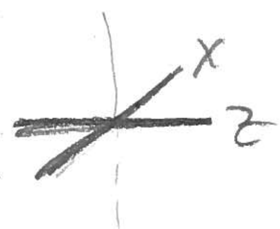
(g) Symmetric and skew-symmetric matrices in $M(n)$ are transverse: let $A \in M(n)$ be given; then $\frac{A^\top + A}{2}$ is symmetric, and $\frac{A^\top - A}{2}$ is skew-symmetric, and their sum is A.

6. *Suppose that X and Z do not intersect transversally in Y. May $X \cap Z$ still be a manifold? If so, must its codimension still be* $\operatorname{codim} X + \operatorname{codim} Z$*? (Can it be?)*

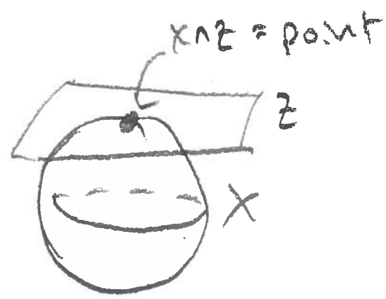
$X \cap Z$ may be a manifold of the expected dimension: for example, let $Y = \mathbb{R}^2$, let X be the x-axis, and let Z be the parabola $y = x^2$.



It may be a manifold of greater-than-expected dimension: for example, let $Y = \mathbb{R}^3$, let X be the x-axis, and let Z be the y-axis.



It may be a manifold of less-than-expected dimension: for example, let $Y = \mathbb{R}^3$, let X be the sphere $x^2 + y^2 + z^2 = 1$, and let Z be the plane $z = 1$.



(Interestingly, this last behavior can only occur with real manifolds, not with complex manifolds: if we write the same equations in $\mathbb{C}^3$, the intersection has the expected dimension, although it is not smooth.)