Solutions to Homework 7

3. *Suppose that Z is a submanifold of X with $\dim Z < \dim X$. Prove that Z has measure zero in X.*

   Let $k = \dim Z$ and $l = \dim X$. First let us prove that $\mathbb{R}^k \subset \mathbb{R}^l$, embedded as $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0 \ldots, 0)$ has measure zero. Let $\epsilon > 0$ be given. For each $n = 1, 2, \ldots$, consider the open box

   $$S_n := (-n, n)^k \times (-\delta, \delta)^{k-l} \subset \mathbb{R}^l,$$

   whose volume is $(2n)^k \cdot (2\delta)^{k-l}$. We want the volume of this box to be $\epsilon/2^n$, so we take

   $$\delta = \tfrac{1}{2} \cdot (\epsilon \cdot 2^{-n} \cdot (2n)^{-k})^{k-l} > 0.$$

   Then $\mathbb{R}^k$ is contained in $S_1 \cup S_2 \cup \cdots$, and the sum of their volumes is $\epsilon/2 + \epsilon/4 + \cdots = \epsilon$, as desired.

   Next we prove the result when $X = \mathbb{R}^l$. Observe that the inclusion $Z \hookrightarrow \mathbb{R}^l$ is an immersion. By the local immersion theorem, for each $z \in Z$ there is a neighborhood $z \in V_z \subset \mathbb{R}^l$ and coordinates $\phi_z \colon V_z \to \mathbb{R}^l$ such that $\phi_z(Z \cap V_z) = \mathbb{R}^k \cap \phi(V_z) \subset \mathbb{R}^l$, which has measure zero as we argued above. Applying ϕ^{-1} and quoting the theorem at the bottom of page 204 in Appendix 1, we see that $Z \cap V_z$ has measure zero in $\mathbb{R}^l$. In lecture we proved that any subset of $\mathbb{R}^l$ is Lindelöf, so there are countably many $z_1, z_2, \ldots$ such that $V_{z_1}, V_{z_2}, \ldots$ cover Z. Thus Z is a countable union of sets of measure zero, hence has measure zero by the argument on page 40.

   Now let X be any l-dimensional manifold. For every local parametrization ψ of X, the preimage $\psi^{-1}(Z)$ is a submanifold of $\mathbb{R}^l$, hence has measure zero in $\mathbb{R}^l$ by the previous paragraph. Thus Z has measure zero in X by the definition in the last paragraph of page 39.

You could have applied the local immersion theorem directly to $Z \subset X$ and used the characterization of measure zero in X later in the same paragraph of page 39 ("It follows that..."), but then if Z is not closed in X you have to worry about finding a parametrization near points $x \in \bar{Z} \setminus Z$.

13. *Show that the determinant function on $M(n)$ is Morse if $n = 2$, but not if $n > 2$.*

First we consider $n = 2$. We have

$$\det \begin{pmatrix} x & y \\ z & w \end{pmatrix} = xw - yz.$$

The derivative of this is

$$d\det_{(x,y,z,w)} = \begin{pmatrix} w & -z & -y & z \end{pmatrix}.$$

Thus the only critical point is the origin, and the Hessian there is

$$\begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

whose determinant is $1 \neq 0$.

For $n \geq 3$, we claim that the origin is a degenerate critical point. We observe that for a matrix

$$\begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix},$$

the determinant is a homogeneous polynomial of degree n in the entries $x_{11}, \dots, x_{nn}$, so its partial derivatives are homogeneous polynomials of degree $n-1$, and its second partials are homogeneous polyomials of degree $n-2$. Thus if $n \geq 2$ then the first partials vanish when $x_{11} = \dots = x_{nn} = 0$, so it is a critical point, and if $n \geq 3$ then the Hessian matrix vanishes at the origin as well, and in particular the determinant of the Hessian is zero, so the origin is a degenerate critical point.

Alternatively, we could remark that the critical points of det are the matrices of rank $\leq n-2$, and that if det is to be a Morse function then the critical points must be isolated, which is not true for $n \geq 3$.

14. The kernel of the linear map $h\colon \mathbb{R}^k \to \mathbb{R}$ given by $h(x_1, \ldots, x_k) = x_k$ is the coordinate plane $\mathbb{R}^{k-1} \oplus 0 \subset \mathbb{R}^k$. The restriction of h to the sphere $S^{k-1} \subset \mathbb{R}^k$ will have a critical point at a point $x \in S^{k-1}$ if and only if the tangent space $T_x(S^{k-1})$ is contained in that kernel; this happens only at the poles $(0, \ldots, 0, \pm 1)$.

    We will show that the north pole is a non-degenerate critical point; the south pole is similar. Let $U \subset \mathbb{R}^{k-1}$ be the open unit disk, and use the parametrization $\psi\colon U \to S^{k-1}$ given by

    $$\psi(x_1, \ldots, x_{k-1}) = (x_1, \ldots, x_{k-1},\, (1 - x_1^2 - \cdots - x_{k-1}^2)^{1/2}).$$

    In these coordinates,

    $$h(x_1, \ldots, x_{k-1}) = (1 - x_1^2 - \cdots - x_{k-1}^2)^{1/2}.$$

    The first derivatives are

    $$\frac{\partial h}{\partial x_i} = -x_i(1 - x_1^2 - \cdots - x_{k-1}^2)^{-1/2}.$$

    Since the $(\ldots)^{-1/2}$ term is positive on U, we see that the only critical point is $(0, \ldots, 0)$. The second derivatives are

    $$\frac{\partial^2 h}{(\partial x_i)^2} = -(1 - x_1^2 - \cdots - x_{k-1}^2)^{-1/2} + x_i^2(1 - x_1^2 - \cdots - x_{k-1}^2)^{-3/2}$$

    and

    $$\frac{\partial^2 h}{\partial x_i \partial x_j} = x_i x_j (1 - x_1^2 - \cdots - x_{k-1}^2)^{-3/2}$$

    if $i \neq j$. Evaluating at the origin, we find that the Hessian matrix is

    $$\begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$$

    which is non-singular.