The Space of Matrices of a Given Rank

Due Friday, February 28, 2020

Fix integers $n \geq m > 0$, and identify $\mathbb{R}^{mn}$ with the space of $m \times n$ matrices, or of linear maps $\mathbb{R}^n \to \mathbb{R}^m$.

A $k \times k$ *minor* of a matrix is the determinant of a $k \times k$ submatrix. The number of $k \times k$ minors of an $m \times n$ matrix is $\binom{m}{k} \cdot \binom{n}{k}$.

1. Show that if some $k \times k$ minor of A does not vanish, then $\operatorname{rank}(A) \geq k$.
2. Show that if $\operatorname{rank}(A) \geq k$ then some $k \times k$ minor of A does not vanish.
3. Show that the set $\Phi_k \subset \mathbb{R}^{mn}$ of matrices of rank $\leq k$ is closed.
4. Show that $\Phi_k \setminus \Phi_{k-1}$ is dense in Φ_k.
5. Show that $\Phi_k \setminus \Phi_{k-1}$ is a manifold of codimension $(m-k)(n-k)$ in $\mathbb{R}^{mn}$.