Solutions to Midterm 1

Each part is worth 5 points, for a total of 40 points.

1. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ and $g\colon \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad\qquad g(x, y, z) = x^2 + y^2 - z^2.$$

On the last homework you showed that the sphere $f^{-1}(a)$ and the hyperboloid $g^{-1}(1)$ are transverse unless $a = 1$. Now consider the function

$$F\colon \mathbb{R}^3 \to \mathbb{R}^2$$

given by

$$F(x, y, z) = (x^2 + y^2 + z^2,\ x^2 + y^2 - z^2),$$

that is, $F = f \times g$.

(a) Find the derivative of F.

Hint: This is a 2×3 matrix whose entries depend on x, y, and z.

Solution:

$$dF_{(x,y,z)} = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ 2z & 2y & -2z \end{pmatrix}$$

(b) Define what it means for a point $(x, y, z) \in \mathbb{R}^3$ to be a *critical point* of F.

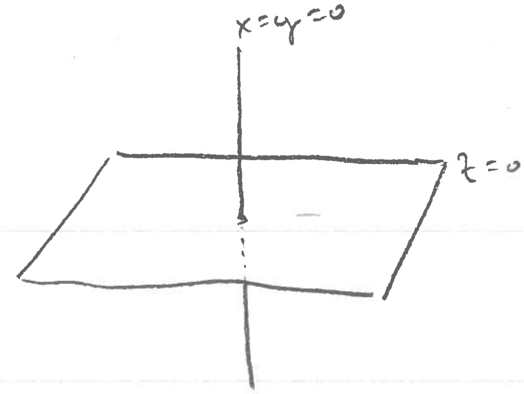
Solution: A point $(x, y, z) \in \mathbb{R}^3$ is a critical point of F if $dF_{(x,y,z)}$ is not surjective, or equivalently, if the two rows of the matrix are linearly dependent.

(c) Describe the set of critical points of F using equations, and draw a picture.

Hint: It's the union of a plane and a line.

Solution: If $z = 0$ then the two rows of dF are the same. If $z \neq 0$ then the two rows are linearly dependent if and only if the second row is -1 times the first, which is true if and only if $x = y = 0$.

So the set of critical points consists of the xy-plane and the z-axis.



(d) Define what it means for a point $(a, b) \in \mathbb{R}^2$ to be a *critical value* of F.

Solution: A point $(a, b) \in \mathbb{R}^2$ is a *critical value* of F if $(a, b) = F(x, y, z)$ for some critical point (x, y, z).

(e) Describe the set of critical values of F using equations, and draw a picture.

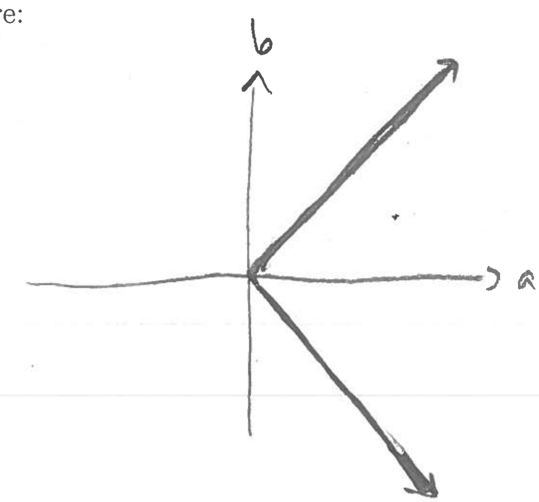
Hint: It's the union of two half-lines.

Sanity check: From the homework you know (or can believe) that $(a, 1)$ is a critical value if and only if $a = 1$.

Solution: If $z = 0$ then $F(x, y, z) = (x^2 + y^2, x^2 + y^2)$, so we can get any point (a, b) with $b = a$ and $a \geq 0$.

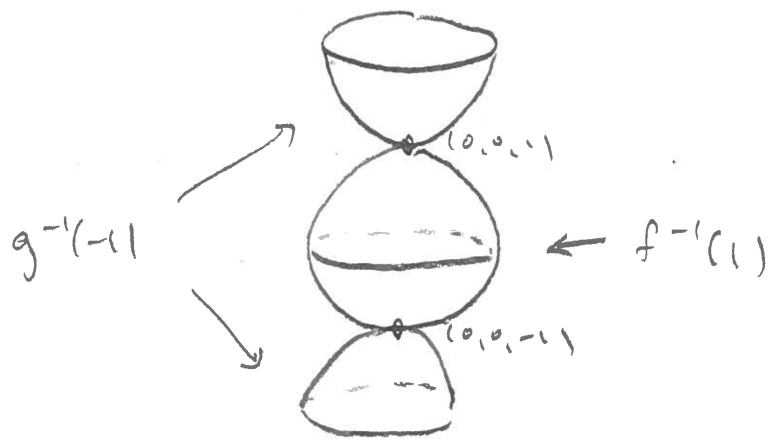
If $x = y = 0$ then $F(x, y, z) = (z^2, -z^2)$, so we can get any point (a, b) with $b = -a$ and $a \geq 0$.

Here is the picture:

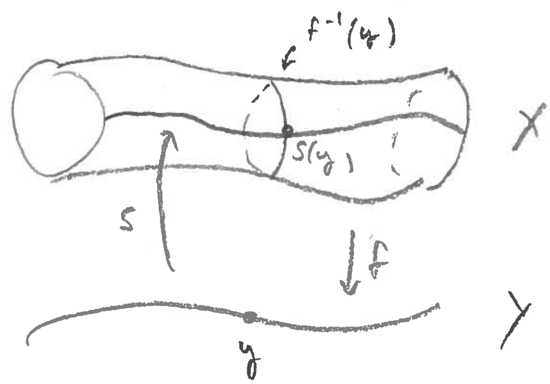


(f) Over the critical value $(1, 1)$, the sphere $f^{-1}(1)$ and hyperboloid $g^{-1}(1)$ are tangent along a circle, as you saw in on the homework. What happens over the critical value $(1, -1)$? Describe it in words and draw a picture.

Solution: The sphere $f^{-1}(1)$ and the hyperboloid of two sheets $g^{-1}(-1)$ are tangent at two points $(0, 0, \pm 1)$:



2. Let $f\colon X \to Y$ be a smooth map between manifolds. A *section* or *cross-section* of f is a map $s\colon Y \to X$ such that $f \circ s = 1_Y$, that is, $f(s(y)) = y$ for all $y \in Y$, or equivalently, $s(y)$ is in the fiber $f^{-1}(y)$ for all $y \in Y$.



(a) Show that if s is a smooth section of f, then for any $y \in Y$ the point $s(y) \in X$ is a regular point of f; or to use another term, f is a submersion at $s(y)$.

Hint: Apply the chain rule to $f \circ s$.

Solution: Following the hint, we note that $f(s(y)) = y$ for all $y \in Y$, so the chain rule applied to $f \circ s$ says that the composition

$$T_y(Y) \xrightarrow{df_y} T_{s(y)}(X) \xrightarrow{df_{s(y)}} T_y(Y)$$

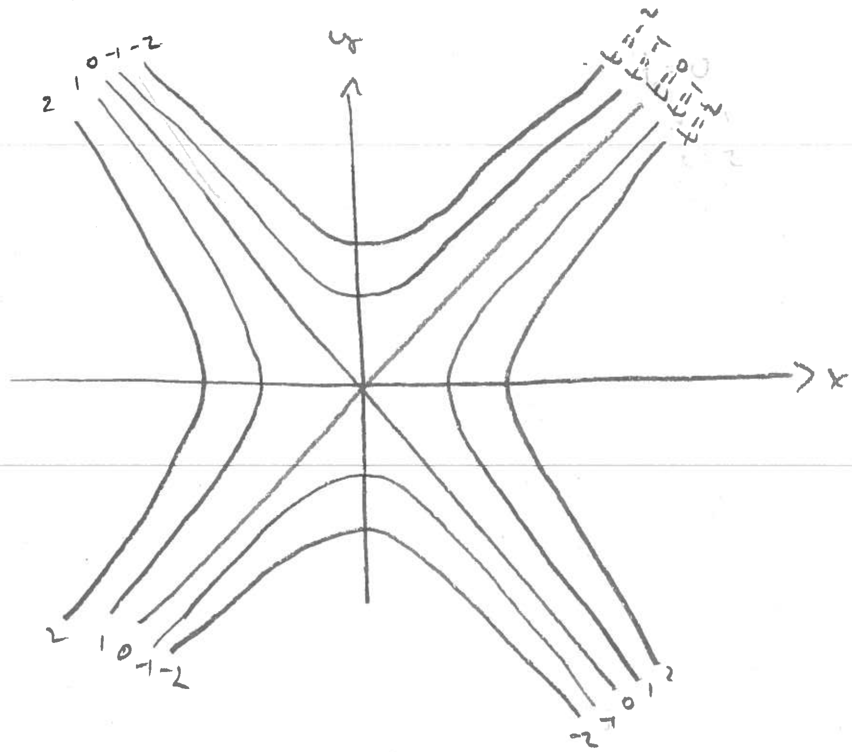
is the derivative of the identity map on Y, which is the identity map on $T_y(Y)$, and in particular it's a surjective map. Thus the second step $df_{s(y)}$ must be surjective, so f is a submersion at $s(y)$.

(b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 - y^2$.

Sketch the fibers $f^{-1}(t)$ for $t = -2, -1, 0, 1, 2$.

The origin $(0, 0) \in \mathbb{R}^2$ is a critical point of f, so by part (a) there is no smooth section $s\colon \mathbb{R} \to \mathbb{R}^2$ of f with $s(0) = (0, 0)$. But exhibit a *continuous* section with $s(0) = (0, 0)$.

Solution: The fibers look like this:



For the continuous section, we could take

$$s(t) = \begin{cases} (\sqrt{t}, 0) & \text{if } t \geq 0 \\ (0, \sqrt{-t}) & \text{if } t \leq 0 \end{cases}$$

We verify that this is a section of f: if $t \geq 0$ then

$$f(s(t)) = f(\sqrt{t}, 0) = (\sqrt{t})^2 = t,$$

and if $t \leq 0$ then

$$f(s(t)) = f(0, \sqrt{-t}) = -(\sqrt{-t})^2 = t$$

as required.